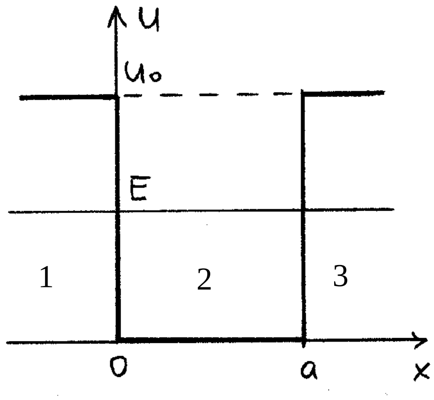
Finite Potential well

Now we deal with the following potential energy



$$U = \begin{cases} 0 & , 0 \le x\ a\ , \\ U_0\ , & x < 0\ ,\ x > a\,. \end{cases}$$

and assume that $E < U_0$, a-width of potential well.

Comparing with $U_0 \to \infty$ case, we are faced with more complicated problem, since wave functions in regions 1 and 3 are nonzero. But the particle can be found in any area with the appropriate probability.

General solutions of corresponding Schrödinger equations for different regions are:

$$\psi_2(x) = B\ e^{ik_2 x} + C\ e^{-\ ik_2 x}\quad ,\quad \psi_1(x) = A\ e^{\kappa_1 x}\ ,\ \ \psi_3(x) = D\ e^{-\ \kappa_1 x}\quad \text{, here}$$

$$k_1 = \frac{\sqrt{2m(U_0 - E)}}{\hbar}, k_2 = \frac{\sqrt{2mE}}{\hbar}\quad .$$

Continuity conditions for $x = 0$ and $x = a$ give

$$A = B + C\quad ;\quad Ak_1 = ik_2(B - C)$$

$$B\,e^{ik_2 a} + Ce^{-ik_2 a} = D\,e^{-ik_1 a}; -\frac{ik_2}{k_1}(Be^{ik_2 a} - Ce^{-ik_2 a}) = De^{-ik_1 a}\quad .$$

I'll start my calculations from the second area. We eliminate A and D, then it reduces to the system for B and C

$$(B + C)k_1 = ik_2(B - C)$$

$$B\,e^{ik_2 a} + Ce^{-ik_2 a} = -\frac{ik_2}{k_1}(B\,e^{ik_2 a} - Ce^{-ik_2 a})\quad .$$

That system has nontrivial solution if the determinant is equal to zero. Writing it as

$$B(k_1 - ik_2) + C(k_1 + ik_2) = 0$$

$$Be^{ik_2 a}(k_1 + ik_2) + Ce^{-ik_2 a}(k_1 - ik_2) = 0\ \ ,$$

we must demand that

$$\begin{vmatrix} k_1 - ik_2 & k_1 + ik_2 \\ e^{i\,k_2\,a}(\kappa_1 + i\ k_2) & e^{-\ i\ k_2\ a}(k_1 - i\ k_2) \end{vmatrix} = 0\ ,$$

which gives

$$(k_1 -\ i\ k_2)^2\ e^{-\ i\ k_2\ a} -\ (k_1 +\ i\ k_2)^2\ e^{i\,k_2\,a} = 0\quad .$$

Real part of above given relation is automatically equal to zero. For the imaginary part we have

$$(k_1^2-k_2^2)\sin k_2 a+2 k_1 k_2 \cos k_2 a=0 \quad ,$$

which is written as

$$\tan k_2 a=\frac{2 k_1 k_2}{k_2^2-k_1^2} \quad .$$

Using the expressions of k_1 and k_2 it may be written as

$$\tan\left(\frac{a\sqrt{2mE}}{\hbar}\right)=\frac{2\sqrt{E(U_0-E)}}{2E-U_0} \quad .$$

This equation can be used to calculate the energy of a particle moving inside a potential well. It is obvious, that this equation is not solvable analytically. It can be solved numerically or graphically. To do this, I want to simplify the last equation a little.

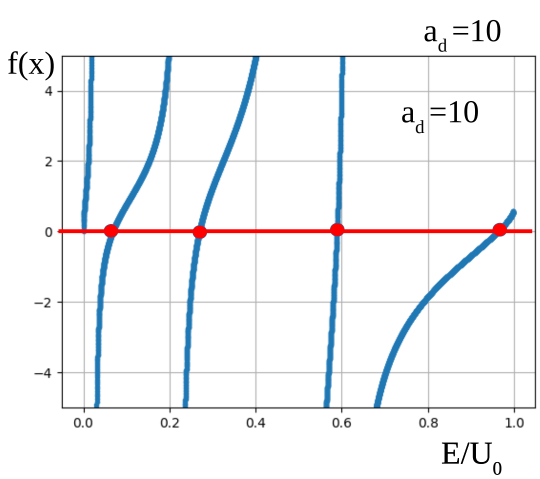
As it was done in the previous section, it makes sense use dimensionless energy $x=\frac{E}{U_0}$ and potential well width $a_d=\frac{a\sqrt{2mU_0}}{\hbar}$, for this new parameters we have:

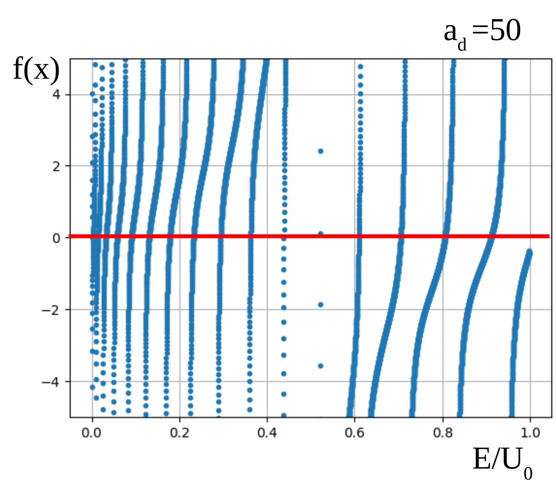
$$\tan(a_d\sqrt{x})=\frac{2\sqrt{x(1-x)}}{2x-1} \text{ , here } \quad 0<x<1.$$

Energies are zeros of the following function:

$$f(x)=\tan(a_d\sqrt{x})-\frac{2\sqrt{x(1-x)}}{2x-1}$$

On the next plots you can see the results of this calculations for two different values for dimensionless width of potential well (the valid energies for particle are marked by red points):





On figures we see that inside the well there is always finite number possible energy states (minimum is 1 state) and it depends how large U_0 is.

The zero energy state should be ignored. **Prove it!**