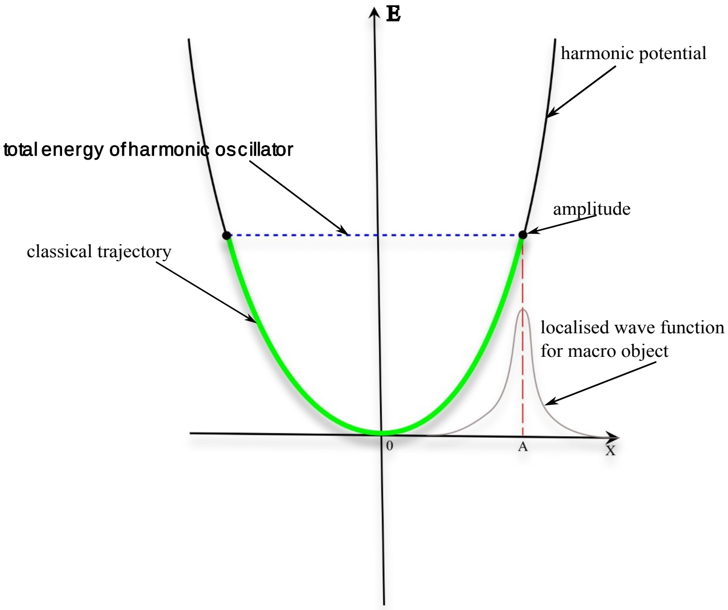
8. Transition from quantum to classical description of the harmonic oscillator.

Classical motion arises as the interference of stationary quantum states.
Klassikaline liikumine tekib statsionaarsete kvantolekute interferentsist.



We have two different solutions that describe the motion of a harmonic oscillator.

The classical solution :

$x(t)=A\cos(\omega t)$

we will use the following classicl initial conditions for x(t=0)=A and p(t=0)=0.

The quantum solution :

$E_n=\hbar\omega(n+\frac{1}{2})$ and wave function $\phi_n(x)$

The question is: how can one obtain a classical time-dependent solution for the coordinate from stationary quantum solutions describing a macroscopic object?This means that the classical solution must emerge as the classical limit of the quantum formalism.

Now we need to define the criteria under which an object can be considered classical.First of all, the vibration amplitude and mass must be large enough (for example, several centimeters and several kilograms). The second criterion is the precise localization of the classical object and that the quantum fluctuations must be neglibally small.

Let us consider a system with vibration amplitude A=1 cm, particle mass m=1 kg , and oscillation frequency ω=10 rad/s (classical object). From the law of energy conservation for this classical oscillator $\frac{1}{2}m\omega^2A^2=\hbar\omega(n+\frac{1}{2})$we obtain the quantum number of the corresponding stationary state $n \approx 10^{31}$. Thus the classical oscillator corresponds to a quantum state involving an enormous number of neighboring stationary states.

In this case the localized time-dependent total wave function $\Psi(x,t)$ can be presented as:

$$\Psi(x,t)=\sum_n C_n\phi(x)e^{-\frac{iE_n}{\hbar}t} \quad (1)$$

here,

$$C_n=\int_{-\infty}^{+\infty}\Psi(x,0)\phi_n^*(x)dx. \quad (2)$$

We assume that initial wavefunction $\Psi(x,0)=\sum_n C_n\phi_n(x)$ localized near the classical oscillation amplitude **A** (see figure and classical initial conditions). It is clear that $\Psi(x,t)$ must be normalized on unit:

$$\int_{-\infty}^{+\infty}\Psi^*(x,t)\Psi(x,t)dx=\int_{-\infty}^{+\infty}\sum_{n,m}C_n^*\phi_n^*(x)e^{-\frac{iE_n}{\hbar}t}C_m\phi_m^*(x)e^{\frac{iE_m}{\hbar}t}dx=\int_{-\infty}^{+\infty}\sum_{n,m}C_n^*C_m\phi_n^*(x)\phi_m(x)e^{\frac{i(E_m-E_n)}{\hbar}t}dx=$$
$$=\sum_{n,m}C_n^*C_me^{\frac{i(E_m-E_n)}{\hbar}t}\int_{-\infty}^{+\infty}\phi_n^*(x)\phi_m(x)dx=\sum_{n,m}C_n^*C_me^{\frac{i(E_m-E_n)}{\hbar}t}\delta_{n,m}=\sum_n C_n^2=1 \quad (3)$$

NB! To undersand the physical meaning of C_n coefficients see Appendix I.

The “classical” motion of this localized quantum wave packet (classical motion of the macro particle) can be described as a the time dependence of the expectation value of the coordinate **<x(t)>** which corresponds to the position of the maximum of the localized wave packet:

$$<x(t)>=\int_{-\infty}^{+\infty}\Psi^*(x,t)x\Psi(x,t)dx=\sum_{n,m}C_m^*C_ne^{\frac{i(E_m-E_n)}{\hbar}t}\int_{-\infty}^{+\infty}\phi_n(x)x\phi_m(x)dx=$$
$$=\sum_n(C_{n+1}^*C_ne^{-i\omega t}+C_n^*C_{n+1}e^{i\omega t})<n|x|n+1>=\{\sum_n 2\,\mathrm{Re}(C_{n+1}^*C_n)<n|x|n+1>\}\cdot\cos(\omega t+\phi_0) \quad (4)$$

Here taken into the account that corresponding integral is equal to:

$$\int_{-\infty}^{+\infty}\phi_n^*(x)x\phi_m(x)dx=\sqrt{\frac{\hbar}{m\omega}}\sqrt{\frac{n+1}{2}}\quad \text{for}\quad n>> \quad => \sqrt{\frac{\hbar n}{2m\omega}} \quad (5)$$

finally we have : $<x(t)>=A\cdot\cos(\omega t)$ here $A=\sum_{n,m}2\,\mathrm{Re}(C_{n+1}^*C_n)<n|x|n+1>$ is an amplitude of vibration.

Some conclusions:

1. Stationary states are time-independent, yet their superposition describes motion.

2. A classical trajectory arises from a superposition of a large number of quantum energy levels. The existence of many such levels produces the time dependence of observables.

3. Classical mechanics emerges because the quantum phases of nonclassical trajectories cancel each other through destructive interference.

Further calculations require specification of the localized wave function.
If we present $\Psi(x,0)$ as high-localized around **A** Gaussian:

$$\Psi(x,0)=\frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}}e^{-\frac{(x-A)^2}{4\sigma^2}} \quad (6)$$

For $\sigma=A/\sqrt{(2)}$ and using (2) we will get for C_n the next expresion:

$$C_n=e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\text{, here }\alpha=\sqrt{\frac{m\omega}{2\hbar}}A. \quad (7)$$

For probability distribution over states we have expression $P_n = C_n^2 = e^{-|\alpha|^2}\frac{\alpha^{2n}}{\sqrt{n!}}$ - this is a Poisson distributon function. The average quantum number can by calculated as a maximum of P_n and equal to $<n> = \alpha^2$. In this case amplitude from equation (7) give: $A = \sqrt{\frac{2\hbar <n>}{m\omega}}$ can be compared with classical amplitude for large values of **n**>>

$$\frac{1}{2} m\omega^2 A^2 = \hbar\omega(n+\frac{1}{2}) => A = \sqrt{\frac{2\hbar n}{m\omega}}. \qquad (8)$$

Thus, because the Planck constant is extremely small, quantum mechanics naturally reproduces the classical solution in the limit of large quantum numbers. Classical motion is the interference pattern of stationary quantum states!

Appendix I.

Any three dimensional vector $\vec{A}$ can be presented in form $\vec{A} = \sum_{k=1}^{3} A_k \vec{e}_k$, here A_k vector coordinates and $\vec{e}_m \equiv \vec{i}, \vec{j}$ and $\vec{k}$ *for* $m = 1..3$ is a base vectors. The projection of $\vec{A}$ on base vector $\vec{e}_m$ can be calculated as scalar product $\vec{A}\vec{e}_m = A_m$.

In quantum mechanics we have essentially the same picture, but for wave functions. We can now use another fundamental property of the eigenfunctions of Hermitian operators: they form a complete set of basis functions in Hilbert space. This means that any wave function can be represented as a linear combination of the eigenfunctions of a Hermitian operator.

For the harmonic oscillator, this means that any wave function $\Psi(x,0)$ can be represented as a vector in an infinite-dimensional Hilbert space (analogical to $\vec{A}$ in 3d space) in form $\Psi(x,0) = \sum_n C_n \phi_n(x)$. Here $\phi_n(x)$ playing a role of basis functions for this space (analogical to $\vec{e}_m$ in 3d space).

The projection of wave function $\Psi(x,0)$ *on basis vector (eigen function of harmonic oscillator)* $\phi_n(x)$ in quantum mechanics can be calculated as follows:

$$\int_{-\infty}^{+\infty} \Psi(x,0)\phi_n^*(x)dx = \int_{-\infty}^{+\infty} (\sum_m C_m \phi_m(x))\phi_n^*(x)dx = C_n \qquad (9)$$

this is means that the C_n is a coordinat of vector $\Psi(x,0)$ in infinite-dimensional Hilbert space.

Expression (3) give us a square of length of vector $\Psi(x,0)$ which equal to $\sum_n C_n^2 = 1$ (total probability to find harmonic oscillator on any state should be equal to one).). Finnaly C_n^2 can be intepreted as a probability to find harmonic oscillator in state with number **n.**