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Mirtel Toming 233465YAFB

20 **6. Show that as $U_0 \to 0$, the transmission coefficient $T(E) \to 1$ for both cases $E > U_0$ and $E < U_0$.**

Case 1: $E > U_0$

The transmission coefficient is given by:

$$T(E) = \frac{4k_1^2k_2^2}{4k_1^2k_2^2\cos^2(k_2a) + (k_1^2 - k_2^2)^2\sin^2(k_2a)}$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}, \quad k_2 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$$

As $U_0 \to 0$, so $k_2 \to k_1$, and thus:

$$k_1^2 - k_2^2 \to 0$$

This simplifies the expression to:

$$T(E) \to \frac{4k_1^4}{4k_1^4} = 1$$

Case 2: $E < U_0$

The transmission coefficient is:

$$T(E) = \frac{4k_1^2k_2^2}{4k_1^2k_2^2\cosh^2(k_2a) + (k_1^2 + k_2^2)^2\sinh^2(k_2a)}$$

where

$$k_2 = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$$

As $U_0 \to 0$, $k_2 \to 0$. Taylor expansions for small arguments:

$$\cosh(k_2a) \approx 1 + \frac{(k_2a)^2}{2}, \quad \sinh(k_2a) \approx k_2a$$

Substituting into the denominator:

$$\cosh^2(k_2a) \approx 1 + (k_2a)^2, \quad \sinh^2(k_2a) \approx (k_2a)^2$$

Thus, the denominator becomes:

$$4k_1^2k_2^2(1 + (k_2a)^2) + (k_1^2 + k_2^2)^2(k_2a)^2$$

As $k_2 \to 0$, the terms involving $(k_2a)^2$ vanish, and it simplifies to:

$$4k_1^2k_2^2$$

So:

$$T(E) \to \frac{4k_1^2k_2^2}{4k_1^2k_2^2} = 1$$

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8. Show that the probability current density (or wavefunction) inside the walls (areas 1 and 3) of an infinite potential well should be zero, and explain why.

The probability current density $J(x,t)$ is defined as:

$$J(x,t) = \frac{\hbar}{2mi}\left(\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x} - \psi(x,t)\frac{\partial\psi^*(x,t)}{\partial x}\right)$$

where:

- $\hbar$ is the reduced Planck constant,
- m is the mass of the particle,
- $\psi(x,t)$ is the wavefunction.

In regions outside the well, the potential $U(x)$ is infinite:

$$U(x) = \infty \quad \text{for} \quad x \leq 0 \quad \text{or} \quad x \geq L$$

the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$$

Substituting $U(x) = \infty$ leads to:

$$\infty \times \psi(x) = E\psi(x) + (\text{finite terms})$$

For the equation to remain finite on both sides, the only solution is:

$$\psi(x) = 0$$

If $\psi(x) \neq 0$, the left side would be infinite, which is impossible. Therefore the probability of finding the particle there is zero.

Since $\psi(x) = 0$ and $\frac{\partial \psi}{\partial x} = 0$, substituting into the expression $J(x,t)$ gives:

$$J(x,t) = 0$$

15 18. Should the wave function inside the walls of a finite potential well be a complex number or a real number? What is your opinion, and why?

The general solution in this region is:

$$\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$$

where

$$\kappa = \sqrt{\frac{2m(U-E)}{\hbar^2}}$$

is a positive real constant since $U > E$ (bound state). Since both $e^{\kappa x}$ and $e^{-\kappa x}$ are real functions, and the coefficients A and B can be taken to be real as well, the wave function $\psi(x)$ itself can be real. Complex exponentials are needed when $U < E$ and the solution is oscillatory.

20 28. How can we calculate the energy of a harmonic oscillator at a non-zero temperature?

Note: Thermal motion is fully chaotic and can be described by a random force acting on an oscillating point mass. Ideas from the calculation of heat capacity in a 1D crystal lattice may be helpful.

The average energy $\bar{E}$ of a harmonic oscillator is:

$$\bar{E} = \hbar\omega \left(\frac{1}{e^{\hbar\omega/kT} - 1} \right)$$

where:

- $\hbar$ is the reduced Planck constant,
- ω is the angular frequency of the oscillator,
- k is the Boltzmann constant,
- T is the absolute temperature.

This result comes from the Bose-Einstein distribution, which gives the average occupation number $\bar{n}$ of the energy levels (for phonons and protons):

$$\bar{n} = \frac{1}{e^{\hbar\omega/kT} - 1}$$

The total thermal energy is then obtained by integrating over all vibrational modes. In 1D crystal models, this leads to:

$$\bar{E}_{\text{tot}} = \frac{2N}{\pi} \int_0^{\omega_0} \frac{\hbar\omega}{\sqrt{\omega_0^2 - \omega^2}} \frac{1}{e^{\hbar\omega/kT} - 1} \, d\omega$$

where:

- N is the total number of atoms,
- ω_0 is the maximum vibrational frequency $\omega_0 = \sqrt{\frac{4g}{m}}$

20 31 Show that the functions $\omega(q) = \omega_0 \cdot \left|\sin\left(\frac{qa}{2}\right)\right|$ and $\omega\left(q + \frac{2\pi}{a}\right)$ describe the same harmonic wave $u_{k,q}(t) = A_q \cdot e^{i(\omega(q)t+qak)}$.(By other words, show that $u_{k,q}(t) = u_{k,q+\frac{2\pi}{a}}(t)$.)

We are asked to show that:

$$u_{k,q}(t) = u_{k,\,q+\frac{2\pi}{a}}(t)$$

where:

$$u_{k,q}(t) = A_q \, e^{i(\omega(q)t+qak)}$$

and

$$\omega(q) = \omega_0 \left|\sin\left(\frac{qa}{2}\right)\right|$$

Substituting $q + \frac{2\pi}{a}$ instead of q, we get:

$$u_{k,\, q+\frac{2\pi}{a}}(t) = A_{q+\frac{2\pi}{a}}\, e^{i(\omega(q+\frac{2\pi}{a})t+(q+\frac{2\pi}{a})ak)}$$

Expanding the phase:

$$= A_{q+\frac{2\pi}{a}}\, e^{i(\omega(q+\frac{2\pi}{a})t+qak+2\pi k)}$$

Since $e^{i2\pi k} = 1$ for any integer k, we simplify:

$$= A_{q+\frac{2\pi}{a}}\, e^{i(\omega(q+\frac{2\pi}{a})t+qak)}$$

Comparing $\omega(q)$ and $\omega\left(q + \frac{2\pi}{a}\right)$

$$\omega\left(q + \frac{2\pi}{a}\right) = \omega_0 \left|\sin\left(\frac{(q+\frac{2\pi}{a})a}{2}\right)\right| = \omega_0 \left|\sin\left(\frac{qa}{2} + \pi\right)\right|$$

Using:

$$\sin(x + \pi) = -\sin(x)$$

Thus:

$$\sin\left(\frac{qa}{2} + \pi\right) = -\sin\left(\frac{qa}{2}\right)$$

Taking the absolute value:

$$\left|\sin\left(\frac{qa}{2} + \pi\right)\right| = \left|-\sin\left(\frac{qa}{2}\right)\right| = \left|\sin\left(\frac{qa}{2}\right)\right|$$

Thus:

$$\omega\left(q + \frac{2\pi}{a}\right) = \omega(q)$$

substituting back, we have:

$$u_{k,\, q+\frac{2\pi}{a}}(t) = A_{q+\frac{2\pi}{a}}\, e^{i(\omega(q)t+qak)}$$

If we choose:

$$A_{q+\frac{2\pi}{a}} = A_q$$

then:

$$u_{k,\, q+\frac{2\pi}{a}}(t) = u_{k,q}(t)$$