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1. How is looks like Schrodinger equation for tunnel effect (topic 4.3) for regions I,II, and III ? Derive equations.

The time-independent Schrödinger equation in one dimension is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x) = E\,\psi(x).$$

For a rectangular barrier of height V_0 extending from $x = 0$ to $x = a$, we split space into three regions:

Region I: $x < 0$, $V(x) = 0$

$$-\frac{\hbar^2}{2m}\psi''(x) = E\,\psi(x) \implies \psi''(x) + k^2\,\psi(x) = 0, \quad k = \frac{\sqrt{2mE}}{\hbar}.$$

General solution:

$$\psi_I(x) = A\,e^{ikx} + B\,e^{-ikx}.$$

Region II: $0 \le x \le a$, $V(x) = V_0$

$$-\frac{\hbar^2}{2m}\psi''(x) + V_0\,\psi(x) = E\,\psi(x) \implies \psi''(x) + q^2\,\psi(x) = 0, \quad q^2 = \frac{2m(E - V_0)}{\hbar^2}.$$

For the tunneling case $E < V_0$, define $\kappa = \sqrt{2m(V_0 - E)}/\hbar$. Then the equation becomes

$$\psi''(x) - \kappa^2\,\psi(x) = 0,$$

with solution

$$\psi_{II}(x) = C\,e^{\kappa x} + D\,e^{-\kappa x}.$$

Region III: $x > a$, $V(x) = 0$ Again

$$\psi''(x) + k^2\,\psi(x) = 0, \quad k = \frac{\sqrt{2mE}}{\hbar}.$$

If we assume only a transmitted wave to the right,

$$\psi_{III}(x) = F\,e^{ikx}.$$

Continuity conditions at $x = 0$ and $x = a$:

$$\begin{array}{ll} \psi_I(0) = \psi_{II}(0): & A + B = C + D, \\ \psi_I'(0) = \psi_{II}'(0): & ik(A - B) = \kappa(C - D), \\ \psi_{II}(a) = \psi_{III}(a): & Ce^{\kappa a} + De^{-\kappa a} = Fe^{ika}, \\ \psi_{II}'(a) = \psi_{III}'(a): & \kappa\big(Ce^{\kappa a} - De^{-\kappa a}\big) = ik\,Fe^{ika}. \end{array}$$

Solving for B and F (with $A = 1$ for convenience) gives

$$B = \frac{(k^2 + \kappa^2)\,\sinh(\kappa a)}{-2\,i\,k\,\kappa\,\cosh(\kappa a) + (k^2 - \kappa^2)\,\sinh(\kappa a)}, F = \frac{2\,i\,k\,\kappa\,e^{-ika}}{2\,i\,k\,\kappa\,\cosh(\kappa a) + (k^2 - \kappa^2)\,\sinh(\kappa a)}$$

Since $F \neq 0$, there is a nonzero transmitted amplitude—i.e. particles have a finite probability to "tunnel through" the barrier.

20 **11. Why the solution for quantum number $n = 0$ must be ignored? Give a physical justification**

The wave functions are

$$\psi_n(x) = A\sin\!\big(n\pi x/L\big),$$

with corresponding energies

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}.$$

Here n is a positive integer.

If we set $n = 0$, then

$$\psi_0(x) = A\sin(0) = 0 \quad \text{everywhere.}$$

That means the wave function is identically zero, so the probability density $|\psi_0(x)|^2$ is zero everywhere. There is no chance of finding the particle anywhere.

Physical justification:

- A wave function must be normalizable and nontrivia so that $\int |\psi|^2 dx = 1$.

- The $n = 0$ solution has $\int 0\,dx = 0$, so it cannot be normalized to unity.
- It carries no probability and hence does not represent a real physical state.

Therefore, we discard the $n = 0$ solution and start with $n = 1, 2, 3, \ldots$, each of which gives a valid, nonzero wavefunction and a real energy.

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18. Should the wave function inside the walls of a finite potential well be a complex number or a real number? What is your opinion, why?

In quantum mechanics the most general wave function $\Psi(x,t)$ is a complex-valued function. However, when we solve the time-independent Schrödinger equation for bound states in a finite potential well, we often write

$$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar},$$

where the spatial part $\psi(x)$ satisfies

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi(x) = E\,\psi(x).$$

Since $V(x)$ is real, this ordinary differential equation has real coefficients. Consequently:

- Any complex solution $\psi(x)$ can be multiplied by a global phase $e^{i\theta}$ without changing physical predictions.
- We are free to choose that global phase so that $\psi(x)$ is *real* everywhere inside and outside the well.

Physical Opinion: Although the full wave function $\Psi(x,t)$ must be complex (to carry time dependence and relative phases), the *spatial* bound-state functions $\psi(x)$ can—and usually are—chosen real. This choice simplifies matching boundary conditions at the walls (ψ and ψ' continuous) without losing any physics, because:

1. The real and imaginary parts of any solution each satisfy the same Schrödinger equation.

2. A real $\psi(x)$ already carries all information about the probability density $|\psi(x)|^2$.
3. There is no net probability current for a stationary bound state, so no phase gradient is needed in $\psi(x)$.

In summary, fundamentally the quantum state is complex, but for bound-state eigenfunctions of a real potential one usually picks the spatial part real for simplicity and clarity.

20 **22. Calculate the mean value of the momentum p_x for the quantum harmonic oscillator. Use tabular integrals from lectures as needed. Compare your result with the corresponding mean momentum for the classical harmonic oscillator.**

The stationary eigenfunctions of the one-dimensional harmonic oscillator are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n\!\left(\sqrt{\tfrac{m\omega}{\hbar}}\, x\right) e^{-\frac{m\omega x^2}{2\hbar}},$$

with energy $E_n = \hbar\omega\,(n + \frac{1}{2})$. The momentum operator is

$$\hat{p}_x = -\,i\hbar\,\frac{d}{dx}.$$

We wish to compute

$$\langle p_x \rangle_n = \int_{-\infty}^{+\infty} \psi_n^*(x)\,\big(-i\hbar\tfrac{d}{dx}\big)\,\psi_n(x)\,dx.$$

1. Notice that for each n, $\psi_n(x)$ is either an even function (if n even) or an odd function (if n odd). Hence

$$\psi_n^*(x)\,\frac{d\psi_n(x)}{dx}$$

is an odd function of x.

2. The integral of any odd function over the symmetric interval $(-\infty, +\infty)$ vanishes:

$$\int_{-\infty}^{+\infty} \big[\mathrm{odd}(x)\big]\,dx = 0.$$

3. Therefore

$$\langle p_x \rangle_n = -\,i\hbar \int_{-\infty}^{+\infty} \psi_n^*(x)\, \frac{d\psi_n(x)}{dx}\, dx = 0.$$

For a classical oscillator of mass m, angular frequency ω, and amplitude A, the position and momentum are

$$x(t) = A\cos(\omega t + \phi), \qquad p(t) = m\,\dot{x}(t) = -\,m\omega A\,\sin(\omega t + \phi).$$

The time-average of $p(t)$ over one period $T = 2\pi/\omega$ is

$$\overline{p} = \frac{1}{T}\int_0^T p(t)\,dt = -\,\frac{m\omega A}{T}\int_0^T \sin(\omega t + \phi)\,dt = 0.$$

Both in the quantum case (for any energy level n) and in the classical case, the average (mean) momentum over a full cycle vanishes:

$$\boxed{\langle p_x \rangle_n = 0, \quad \overline{p} = 0.}$$

This reflects the symmetry of the motion: there is no net drift in either direction over one full oscillation.

20 **37. Can you show that (for a 1D crystal) the phase and group velocities of harmonic waves become equal in the limit of very long wavelengths ($\lambda \to \infty$)?**

Consider a one-dimensional monatomic crystal of atoms of mass M connected by springs of constant C, with lattice constant a. The dispersion relation for small oscillations is

$$\omega(k) \;=\; 2\sqrt{\frac{C}{M}}\,\left|\sin\!\left(\tfrac{ka}{2}\right)\right|.$$

The phase velocity is

$$v_{\mathrm{ph}}(k) \;=\; \frac{\omega(k)}{k} \;=\; \frac{2}{k}\sqrt{\frac{C}{M}}\,\left|\sin\!\left(\tfrac{ka}{2}\right)\right|,$$

and the group velocity is

$$v_{\mathrm{g}}(k) \;=\; \frac{d\omega}{dk} \;=\; 2\sqrt{\frac{C}{M}}\,\frac{d}{dk}\left|\sin\!\left(\tfrac{ka}{2}\right)\right| \;=\; a\sqrt{\frac{C}{M}}\,\cos\!\left(\tfrac{ka}{2}\right).$$

In the long-wavelength limit $k \to 0$ (so $\lambda = 2\pi/k \to \infty$), we use the small-angle approximations

$$\sin\!\left(\tfrac{ka}{2}\right) \approx \tfrac{ka}{2}, \qquad \cos\!\left(\tfrac{ka}{2}\right) \approx 1.$$

Substituting into the velocities gives

$$v_{\mathrm{ph}}(k) \approx \frac{2}{k}\sqrt{\frac{C}{M}}\,\frac{ka}{2} = a\sqrt{\frac{C}{M}},$$

$$v_{\mathrm{g}}(k) \approx a\sqrt{\frac{C}{M}}\,\times 1 = a\sqrt{\frac{C}{M}}.$$

Hence

$$\boxed{v_{\mathrm{ph}}(k) \;\to\; a\sqrt{\tfrac{C}{M}} \quad \text{and} \quad v_{\mathrm{g}}(k) \;\to\; a\sqrt{\tfrac{C}{M}} \quad \text{as } k \to 0,}$$

showing that the phase and group velocities coincide for very long wavelengths.