Loengud 5-9

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Question 1

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What is the physical interpretation and units of the wave function in 1D, 2D, and 3D?

The wave function $\Psi(\mathbf{r}, t)$ describes the quantum state of a particle. Its squared modulus, $|\Psi(\mathbf{r}, t)|^2$, represents the probability density of finding a particle at a given position and time.

For different spatial dimensions, the wave function follows:

- **1D:** $\Psi(x, t)$, probability $dP = |\Psi(x, t)|^2 dx$, units: $\mathrm{m}^{-1/2}$.
- **2D:** $\Psi(x, y, t)$, probability $dP = |\Psi(x, y, t)|^2 dxdy$, units: m^{-1}.
- **3D:** $\Psi(x, y, z, t)$, probability $dP = |\Psi(x, y, z, t)|^2 dxdydz$, units: $\mathrm{m}^{-3/2}$.

These units ensure that the probability remains dimensionless when integrated over space.

Question 15

10

Given the eigenvalue problem for the Hamiltonian operator:

$$\hat{H}\psi_n = E_n\psi_n, \tag{1}$$

where ψ_n is a set of orthonormal eigenfunctions of $\hat{H}$, express the general solution for an arbitrary state function ϕ as a linear combination of eigenfunctions. How can the coefficients c_n be calculated?

Answer

Since the set of eigenfunctions $\{\psi_n\}$ forms a complete basis, any state function ϕ can be expressed as:

$$\phi = \sum_n c_n\psi_n. \tag{2}$$

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To find the expansion coefficients c_n, we take the inner product with ψ_n:

$$c_n = \langle \psi_n | \phi \rangle = \int \psi_n^*(x) \phi(x) \, dx. \tag{3}$$

These coefficients represent the projection of ϕ onto the eigenfunctions ψ_n and determine the contribution of each eigenstate to ϕ.

Question 24 20

Calculate the average ($\langle x \rangle_n, \langle p \rangle_n$) and exact ($x_n, p_n$) values for a free particle in 1D. Are there any issues with these calculations? If so, why?

Answer

For a free particle in 1D, the wave functions are typically plane waves:

$$\psi_n(x) = \frac{1}{\sqrt{L}} e^{ik_n x}, \tag{4}$$

where $k_n = \frac{2\pi n}{L}$. The expectation values are:

$$\langle x \rangle_n = \int x |\psi_n(x)|^2 \, dx, \tag{5}$$

which is undefined for a free particle due to uniform probability distribution over all space. The momentum expectation value is:

$$\langle p \rangle_n = \int \psi_n^* \hat{p} \psi_n \, dx = \hbar k_n. \tag{6}$$

Exact values of position and momentum cannot be simultaneously well-defined due to the Heisenberg uncertainty principle. The issue arises because the position expectation value is undefined, and a plane wave does not correspond to a localized particle.

Question 30 20

Is there a fundamental limitation on the lower limit of the error of physical measurements in classical physics?

Answer

In classical physics, there is no fundamental lower limit to measurement errors. Theoretically, measurements can be made arbitrarily precise, limited only by technological constraints, experimental imperfections, and external disturbances. Unlike quantum mechanics, where the Heisenberg uncertainty principle imposes a fundamental limit on precision, classical physics assumes that with perfect instruments and conditions, exact values of physical quantities can be determined.

Question 32

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Show that for a potential barrier with $E > U_0$, the flux of incident particles (probability density current) moving towards the barrier is given by:

$$j_i = \frac{\hbar k_1 A^2}{m}, \quad \text{where} \quad k_1 = \frac{\sqrt{2mE}}{\hbar}. \tag{7}$$

Answer

The probability current density is defined as:

$$j = \frac{\hbar}{2mi}\left(\Psi^* \frac{d\Psi}{dx} - \Psi \frac{d\Psi^*}{dx}\right). \tag{8}$$

For a plane wave solution describing an incident particle moving towards the barrier:

$$\Psi(x) = Ae^{ik_1 x}, \tag{9}$$

where $k_1 = \frac{\sqrt{2mE}}{\hbar}$.

Substituting this into the probability current density equation:

$$j_i = \frac{\hbar}{2mi}\left(A^* e^{-ik_1 x} ik_1 A e^{ik_1 x} - Ae^{ik_1 x}(-ik_1 A^* e^{-ik_1 x})\right). \tag{10}$$

Simplifying,

$$j_i = \frac{\hbar k_1}{m}|A|^2. \tag{11}$$

Thus, we have derived the required expression for the flux of incident particles.