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8. Can you show that the average value for some physical quantity A (measurable value for given quantity) does not depend on time for stationary solution of full Schrodinger equation

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A stationary solution of the Schrödinger equation can be written as:

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-iEt/\hbar},$$

The expected value of $\hat{A}$ at time t is given by:

$$\langle A\rangle(t) = \int \psi^*(\mathbf{r}, t)\, \hat{A}\, \psi(\mathbf{r}, t)\, dV.$$

Substituting:

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-iEt/\hbar}, \quad \psi^*(\mathbf{r}, t) = \phi^*(\mathbf{r})e^{iEt/\hbar},$$

we get:

$$\langle A\rangle(t) = \int \phi^*(\mathbf{r})Ae^{iEt/\hbar}\, \hat{A}\, \phi(\mathbf{r})e^{-iEt/\hbar}\, dV.$$

The exponential terms cancel:

$$\langle A\rangle(t) = \int \phi^*(\mathbf{r})\hat{A}\phi(\mathbf{r})\, dV.$$

and we get an equation that isn't dependant on time.

13. Prove that kinetic energy operator is Hermitian

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The kinetic energy operator is:

$$\hat{T} = -\frac{\hbar^2}{2m}\nabla^2,$$

where ∇^2 is the Laplacian operator.

We must prove that

$$\langle\phi|\hat{T}\psi\rangle = \langle\hat{T}\phi|\psi\rangle.$$

Left side:

$$\langle\phi|\hat{T}\psi\rangle = \int \phi^*(\vec{r}) \left(-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r})\right) d^3r.$$

$$= -\frac{\hbar^2}{2m}\int \phi^*(\vec{r})\nabla^2\psi(\vec{r})\,d^3r.$$

$$\int \phi^*\nabla^2\psi\,d^3r = \int \nabla\phi^*\cdot\nabla\psi\,d^3r.$$

?

So:

$$\langle\phi|\hat{T}\psi\rangle = -\frac{\hbar^2}{2m}\int \nabla\phi^*(\vec{r})\cdot\nabla\psi(\vec{r})\,d^3r.$$

Right side:

$$\langle\hat{T}\phi|\psi\rangle = \int \left(-\frac{\hbar^2}{2m}\nabla^2\phi(\vec{r})\right)^*\psi(\vec{r})\,d^3r = -\frac{\hbar^2}{2m}\int \nabla^2\phi^*(\vec{r})\psi(\vec{r})\,d^3r.$$

$$\int \nabla^2\phi^*(\vec{r})\,\psi(\vec{r})\,d^3r = \int \nabla\phi^*(\vec{r})\cdot\nabla\psi(\vec{r})\,d^3r.$$

?

$$\langle\hat{T}\phi|\psi\rangle = -\frac{\hbar^2}{2m}\int \nabla\phi^*(\vec{r})\cdot\nabla\psi(\vec{r})\,d^3r.$$

Since:

$$\langle\phi|\hat{T}\psi\rangle = \langle\hat{T}\phi|\psi\rangle,$$

the kinetic energy operator is Hermitian.

22. Calculate the probability current density function for periodic motion of free particle in one-dimensional space. Give a physical interpretation of the result obtained.

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free particle with momentum p:

$$\Psi(x,t) = Ae^{-\frac{i}{\hbar}(Et-px)}.$$

the probability density is

$$\rho = |\Psi|^2 = |A|^2 \quad ,$$

and the probability current density is

$$j = \frac{i\hbar}{2M}\left(\frac{d\Psi^*}{dx}\Psi - \Psi^*\frac{d\Psi}{dx}\right).$$

derivatives:

$$\frac{d\Psi}{dx} = -\frac{ip}{\hbar}\Psi, \qquad \frac{d\Psi^*}{dx} = \frac{ip}{\hbar}\Psi^*.$$

Substitute:

$$j = \frac{i\hbar}{2M}\left(\frac{ip}{\hbar}\Psi^*\Psi + \frac{ip}{\hbar}\Psi\Psi^*\right) = \frac{p}{M}|\Psi|^2 = \frac{p}{M}|A|^2.$$

Since $p = Mv$, we get:

$$j = v|A|^2,$$

which describes the particle moving with velocity v, or the flux of moving particles. The rate of flow of probability is proportional to the velocity of the particle.

31. How is the Heisenberg uncertainty principle related to the error of physical measurements?

The Heisenberg uncertainty principle is expressed as: 15

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2},$$

this is not a general expression

where Δx and Δp are the uncertainties in position and momentum. The more precisely we measure the position x of a particle, the less precisely we can know its momentum p, and vice versa. Even with perfect measuring devices, it is impossible to simultaneously determine both position and momentum of quantum objects.

33. Show that for a potential barrier where the total energy E is greater than the barrier height U_0, the flux for reflected particles (reflected from the barrier) is given by:

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$$J_R = \frac{\hbar k_1 |B|^2}{m}, \tag{1}$$

where

$$k_1 = \frac{\sqrt{2mE}}{\hbar}. \tag{2}$$

The reflected wavefunction is:

$$\psi_R(x) = Be^{-ik_1 x}, \quad \psi_R^*(x) = B^* e^{ik_1 x}.$$

derivatives:

$$\frac{d\psi_R}{dx} = -ik_1 Be^{-ik_1 x}, \quad \frac{d\psi_R^*}{dx} = ik_1 B^* e^{ik_1 x}.$$

probability current density:

$$j = \frac{i\hbar}{2m}\left(\frac{d\psi^*}{dx}\psi - \psi^*\frac{d\psi}{dx}\right).$$

substitute:

$$\begin{aligned}
j_R &= \frac{i\hbar}{2m}\left(ik_1 B^* e^{ik_1 x} \cdot B e^{-ik_1 x} - B^* e^{ik_1 x} \cdot (-ik_1 B e^{-ik_1 x})\right) \\
&= \frac{i\hbar}{2m}\left(ik_1 |B|^2 + ik_1 |B|^2\right) \\
&= \frac{i\hbar}{2m}(2ik_1 |B|^2) \\
&= \frac{\hbar k_1}{m}|B|^2
\end{aligned}$$

So the flux of reflected particles is:

$$j_R = \frac{\hbar k_1}{m}|B|^2,$$

where $k_1 = \frac{\sqrt{2mE}}{\hbar}$.