85

1 Küsimus 9:

20

$$\psi = c_1\psi_1 + c_2\psi_2 \tag{1}$$

$$\langle E \rangle = \frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle} \tag{2}$$

Using the linearity of the Hamiltonian operator:

$$\hat{H}\psi = \hat{H}(c_1\psi_1 + c_2\psi_2) = c_1\hat{H}\psi_1 + c_2\hat{H}\psi_2 \tag{3}$$

Since ψ_1 and ψ_2 are eigenfunctions of $\hat{H}$ with eigenvalues E_1 and E_2, we substitute:

$$\hat{H}\psi_1 = E_1\psi_1, \quad \hat{H}\psi_2 = E_2\psi_2 \tag{4}$$

$$\hat{H}\psi = c_1E_1\psi_1 + c_2E_2\psi_2 \tag{5}$$

Calculate $\langle \psi | \hat{H} | \psi \rangle$

$$\langle \psi | \hat{H} | \psi \rangle = \langle (c_1\psi_1 + c_2\psi_2) | (c_1E_1\psi_1 + c_2E_2\psi_2) \rangle \tag{6}$$

Expanding:

$$= c_1^*c_1E_1\langle \psi_1|\psi_1\rangle + c_1^*c_2E_2\langle \psi_1|\psi_2\rangle + c_2^*c_1E_1\langle \psi_2|\psi_1\rangle + c_2^*c_2E_2\langle \psi_2|\psi_2\rangle \tag{7}$$

Since ψ_1 and ψ_2 are orthonormal:

$$\langle \psi_1|\psi_1\rangle = 1, \quad \langle \psi_2|\psi_2\rangle = 1, \quad \langle \psi_1|\psi_2\rangle = 0 \tag{8}$$

$$\langle \psi | \hat{H} | \psi \rangle = c_1^*c_1E_1 + c_2^*c_2E_2 \tag{9}$$

Calculate $\langle \psi | \psi \rangle$

$$\langle \psi | \psi \rangle = \langle c_1\psi_1 + c_2\psi_2 | c_1\psi_1 + c_2\psi_2 \rangle \tag{10}$$

$$= c_1^*c_1\langle \psi_1|\psi_1\rangle + c_1^*c_2\langle \psi_1|\psi_2\rangle + c_2^*c_1\langle \psi_2|\psi_1\rangle + c_2^*c_2\langle \psi_2|\psi_2\rangle \tag{11}$$

Since ψ_1 and ψ_2 are orthonormal:

$$\langle \psi | \psi \rangle = c_1^*c_1 + c_2^*c_2 \tag{12}$$

Calculate the Expectation Value of Energy

$$\langle E \rangle = \frac{c_1^*c_1E_1 + c_2^*c_2E_2}{c_1^*c_1 + c_2^*c_2} \tag{13}$$

Condition on coefficients c_1 and c_2 For ψ to be a valid quantum state, it must be normalized:

$$\langle \psi | \psi \rangle = 1 \tag{14}$$

Thus, the coefficients must satisfy:

$$|c_1|^2 + |c_2|^2 = 1 \tag{15}$$

2 Küsimus 13:

5 kus on vahevalemid ?

The kinetic energy operator is given by:

$$\hat{T} = \frac{-\hbar^2}{2m}\nabla^2 \tag{16}$$

where ∇^2 is the Laplacian operator.

An operator $\hat{A}$ is Hermitian if it satisfies the condition:

$$\langle \phi | \hat{A}\psi \rangle = \langle \hat{A}\phi | \psi \rangle \tag{17}$$

for all well-behaved functions ϕ and ψ.

To check whether $\hat{T}$ is Hermitian, we evaluate the inner product:

$$I = \int \phi^*(\mathbf{r})\hat{T}\psi(\mathbf{r})\, d^3r. \tag{18}$$

Substituting $\hat{T}$:

$$I = \int \phi^*(\mathbf{r}) \left(\frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) \right) d^3r. \tag{19}$$

Using integration by parts and assuming that ϕ and ψ vanish at infinity, the boundary terms disappear, leaving:

$$I = \int \left(\frac{-\hbar^2}{2m}\nabla^2\phi^*(\mathbf{r}) \right) \psi(\mathbf{r}) d^3r. \tag{20}$$

Rewriting:

$$I = \int \left(\hat{T}\phi(\mathbf{r}) \right)^* \psi(\mathbf{r}) d^3r. \tag{21}$$

This shows that:

$$\langle \phi | \hat{T}\psi \rangle = \langle \hat{T}\phi | \psi \rangle, \tag{22}$$

which confirms that $\hat{T}$ is Hermitian.

3 Küsimus 16:

20

The momentum operator in the x-direction is given by:

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}. \tag{23}$$

The eigenvalue equation for the momentum operator is:

$$\hat{p}_x\psi_n = p_n\psi_n, \tag{24}$$

where p_n is the eigenvalue corresponding to the eigenfunction ψ_n.

The expectation value of momentum for a state ψ is given by:

$$\langle p \rangle_n = \int \psi_n^* \hat{p}_x \psi_n \, dv. \tag{25}$$

Substituting $\hat{p}_x \psi_n = p_n \psi_n$:

$$\langle p \rangle_n = \int \psi_n^* p_n \psi_n \, dv. \tag{26}$$

Since p_n is a constant, it can be factored out:

$$\langle p \rangle_n = p_n \int \psi_n^* \psi_n \, dv. \tag{27}$$

For a normalized wavefunction, $\int |\psi_n|^2 \, dv = 1$, so we obtain:

$$\langle p \rangle_n = p_n. \tag{28}$$

Thus, if ψ_n is a pure eigenstate of $\hat{p}_x$, then the expectation value $\langle p \rangle_n$ is equal to p_n.

However, if the wavefunction is a superposition of multiple eigenstates:

$$\psi = \sum_n c_n \psi_n, \tag{29}$$

then the expectation value of momentum is given by:

$$\langle p \rangle = \sum_n |c_n|^2 p_n, \tag{30}$$

which is a weighted sum of eigenvalues, not necessarily equal to any single eigenvalue.

The expectation value $\langle p \rangle_n = p_n$ holds only if the system is in a pure eigenstate of the momentum operator. If the state is a superposition of eigenstates, the expectation value is generally different from any individual eigenvalue.

4 Küsimus 26: 20

The Heisenberg uncertainty principle states that the product of the uncertainties in position (Δx) and momentum (Δp) is bounded by:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} \tag{31}$$

Derivation

We derive this using the Schwarz inequality and the properties of quantum operators.

Step 1:

The position $\hat{x}$ and momentum $\hat{p}$ operators satisfy the canonical commutation relation:

$$[\hat{x}, \hat{p}] = i\hbar \tag{32}$$

where $\hat{p} = -i\hbar\frac{d}{dx}$ in the position representation.

Step 2:

For an observable $\hat{A}$, its expectation value in a state $|\psi\rangle$ is given by:

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle \tag{33}$$

The uncertainty in $\hat{A}$ is:

$$\Delta A = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2} \tag{34}$$

For position and momentum:

$$\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}, \quad \Delta p = \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2} \tag{35}$$

Deviation operators:

$$\hat{A} = \hat{x} - \langle\hat{x}\rangle, \quad \hat{B} = \hat{p} - \langle\hat{p}\rangle \tag{36}$$

Step 3:

The Schwarz inequality states:

$$\langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle\langle\psi|\hat{B}^\dagger\hat{B}|\psi\rangle \geq \left|\langle\psi|\hat{A}\hat{B}|\psi\rangle\right|^2 \tag{37}$$

Expanding in terms of variances:

$$(\Delta x)^2(\Delta p)^2 \geq \left|\frac{1}{2}\langle[\hat{x}, \hat{p}]\rangle\right|^2 \tag{38}$$

Using the commutator relation $[\hat{x}, \hat{p}] = i\hbar$, we get:

$$\left|\frac{1}{2}i\hbar\right|^2 = \frac{\hbar^2}{4} \tag{39}$$

We obtain the final result:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} \tag{40}$$

5 Küsimus 32:

20

For a step potential where $E > U_0$, the flux of incident particles (probability current density) is given by:

$$j_i = \frac{\hbar k_1 A^2}{m} \tag{41}$$

where

$$k_1 = \frac{\sqrt{2mE}}{\hbar} \tag{42}$$

Step 1: Define the Wavefunction

In the region before the barrier (Region I), the wavefunction can be written as:

$$\psi_1(x) = Ae^{ik_1 x} + Be^{-ik_1 x} \tag{43}$$

Step 2: Probability Current Density Definition

The probability current density $j(x)$ is defined as:

$$j = \frac{\hbar}{2mi}\left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right) \tag{44}$$

For the incident wave $\psi_i(x) = Ae^{ik_1 x}$, we compute its derivative:

$$\frac{d\psi_i}{dx} = ik_1 Ae^{ik_1 x} \tag{45}$$

$$\frac{d\psi_i^*}{dx} = -ik_1 A^* e^{-ik_1 x} \tag{46}$$

Step 3: Compute Incident Flux j_i

Substituting into the probability current density equation:

$$\begin{aligned} j_i &= \frac{\hbar}{2mi}\left(A^* e^{-ik_1 x} \cdot ik_1 Ae^{ik_1 x} - Ae^{ik_1 x} \cdot (-ik_1 A^* e^{-ik_1 x})\right) && (47) \\ &= \frac{\hbar}{2mi}\left(ik_1 A^* A - (-ik_1 AA^*)\right) && (48) \\ &= \frac{\hbar}{2mi}\left(ik_1 |A|^2 + ik_1 |A|^2\right) && (49) \\ &= \frac{\hbar}{m} k_1 |A|^2 && (50) \end{aligned}$$

We obtain:

$$j_i = \frac{\hbar k_1}{m} A^2 \tag{51}$$