34. Proof that for reflection coefficient $\lim\limits_{E \to \infty} R(x) = 0$

$$\lim_{E \to \infty} R(E) = \left(\frac{\sqrt{E} - \sqrt{E-1}}{\sqrt{E} + \sqrt{E+1}} \right)^2 = \left(\frac{\lim_{E \to 0} \left(\sqrt{E} - \sqrt{E-1} \right)}{\lim_{E \to \infty} \left(\sqrt{E} + \sqrt{E+1} \right)} \right)^2 =$$

$$= \left(\frac{\lim_{E \to \infty} \left(\frac{(\sqrt{E} - \sqrt{E-1})(\sqrt{E} + \sqrt{E-1})}{\sqrt{E} + \sqrt{E-1}} \right)}{\lim_{E \to \infty} \left(\sqrt{E} + \sqrt{E+1} \right)} \right)^2 = \left(\frac{\lim_{E \to \infty} \left(\frac{E - (E-1)}{\sqrt{E} + \sqrt{E+1}} \right)}{\lim_{E \to \infty} \left(\sqrt{E} + \sqrt{E+1} \right)} \right)^2 =$$

$$= \left(\frac{\overbrace{\lim_{E \to \infty} \left(\frac{1}{\sqrt{E} + \sqrt{E+1}} \right)}^{\frac{1}{\infty} = 0}}{\underbrace{\lim_{E \to \infty} \left(\sqrt{E} + \sqrt{E+1} \right)}_{\infty}} \right)^2 = \left(\frac{0}{\infty} \right)^2 = 0$$

26. Derive Heisenberg uncertainty principle.

Consider a system in a quantum mechanical state $|\psi\rangle$ and imagine measuring 2 observables represented by Hermitian operators A and B. For simplicity, we will assume that the spectra of both operators are discrete with $A|a_i\rangle = a_i|a_i\rangle$ and $B|b_j\rangle = b_j|b_j\rangle$

The measurement of A yields as the result one of the eigenvalues a_i and if we expand the state of the system in terms of eigenkets of A (they form a complete orthonormal set) we have $|\psi\rangle = \sum_i \alpha_i |\alpha_i\rangle$, where $\alpha_i = \langle\psi|a_i\rangle$

The results of measurement A yields a distribution of values with the mean given by $\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_i |\alpha_i|^2 a_i$

Similarly, one can derive $\langle A^2\rangle$ and this enables one to define σ_A^2 for this state for the observable A as usual by

$\sigma_A^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$. This is defined to be the uncertainty in the measurements of A.

We will derive a constraint on a simultaneous measurement of 2 observables, by comparing the uncertainties in measuring A and B on the same state $|\psi\rangle$

Define $\Delta A = A - \langle A\rangle$ and $\Delta B = B - \langle B\rangle$, where $\langle A\rangle =$
$= \langle\psi|A|\psi\rangle$.

if $|\phi\rangle = (\Delta A - i\lambda\Delta B)|\psi\rangle$ with real λ, $\langle\phi|\phi\rangle \geq 0$

We write out in detail noting the Hermitian nature of ΔA and ΔB:

$- \langle\psi|(\Delta A + i\lambda\Delta B)(\Delta A - i\lambda\Delta B)|\psi\rangle \geq 0$

since we have $\sigma_A^2 = \langle\psi|(\Delta A)^2|\psi\rangle$ and $\sigma_B^2 = \langle\psi|(\Delta B)^2|\psi\rangle$

We obtain the following condition upon substitution

$\sigma_A^2 + \lambda^2\sigma_B^2 - i\lambda\langle\psi|[\Delta A, \Delta B]|\psi\rangle \geq 0$

$[\Delta A, \Delta B] = [A, B]$ is clear from definitions of ΔA and ΔB.

$-i[A,B]$ - Hermitian operator (lets call it C) and so $-i\langle[A,B]\rangle$ is real,

We rewrite earlier results as $\lambda^2\sigma_B^2 + \lambda\langle C\rangle + \sigma_A^2 \geq 0$, we get $\langle C\rangle^2 \leq 4\sigma_A^2\sigma_B^2$ $\quad (ax^2 + bx + c = 0 \Rightarrow b^2 - 4ac < 0)$.

We substitute C and get $\sigma_A^2\sigma_B^2 \geq \langle \frac{1}{2i}[A,B]\rangle^2$.

For $A = x$ and $B = p_x$, we get $\sigma_x \sigma_p \geq \frac{\hbar}{2}$, a standard form of Heisenberg principle.

19. In one-dimensional periodic system, such as a free particle on a ring or a particle in a 1D box with periodic boundary conditions, the quantum number n can take both positive and negative integer values.

These quantum numbers correspond to discrete momentum states due to the wave nature of the particle and the boundary condition $\psi(x+L) = \psi(x)$, where L is the period (length of the loop or box)

- Positive values of n - motion in one direction; represent positive momentum states.
- Negative values of n - motion in the opposite direction; represent negative momentum states

Thus, the sign of n reflects direction of motion for particle.

- $n = -2 \rightarrow$ General solution for wavefunction of free particle with periodic boundary conditions is $\psi_n(x) = \frac{1}{\sqrt{L}} e^{i\frac{2\pi n}{L}x}$

So for $n = -2 \rightarrow \psi_{-2}(x) = \frac{1}{\sqrt{L}} e^{-i\frac{4\pi}{L}x}$

Momentum: $p_n = \hbar k_n = \hbar \cdot \frac{2\pi n}{L} \rightarrow p_{-2} = \hbar \cdot \left(\frac{-4\pi}{L}\right) = -\frac{4\pi\hbar}{L}$

Energy: $E_n = \frac{p_n^2}{2m} = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2\pi n}{L}\right)^2 \Rightarrow E_{-2} = \frac{\hbar^2}{2m}\left(\frac{4\pi}{L}\right)^2 = \frac{8\pi^2\hbar^2}{mL^2}$

9. Given: $\hat{H}\psi_1 = E_1\psi_1$, $\hat{H}\psi_2 = E_2\psi_2$, linear combination: $\psi = c_1\psi_1 + c_2\psi_2$

What is energy in state ψ: $E = \langle\psi|\hat{H}|\psi\rangle$

Substitute $\psi = c_1\psi_1 + c_2\psi_2$ $\rightarrow$ $E = \langle c_1\psi_1 + c_2\psi_2 | \hat{H} | c_1\psi_1 + c_2\psi_2\rangle$

Using linearity of inner product and Hermitian operator:

$$E = |c_1|^2\langle\psi_1|\hat{H}|\psi_1\rangle + |c_2|^2\langle\psi_2|\hat{H}|\psi_2\rangle + \underbrace{c_1^*c_2\langle\psi_1|\hat{H}|\psi_2\rangle}_{0} + \underbrace{c_2^*c_1\langle\psi_2|\hat{H}|\psi_1\rangle}_{0}$$

Since: $\hat{H}\psi_1 = E_1\psi_1$, so $\langle\psi_1|\hat{H}|\psi_1\rangle = E_1$ and ψ_1 and ψ_2 are $\perp$ $\rightarrow$ orthonormal

$\hat{H}\psi_2 = E_2\psi_2$, so $\langle\psi_2|\hat{H}|\psi_2\rangle = E_2$

$\rightarrow$ the cross term vanishes: $\langle\psi_1|\hat{H}|\psi_2\rangle = E_2\langle\psi_1|\psi_2\rangle = 0$ $\Rightarrow$ So we

$\langle\psi_2|\hat{H}|\psi_1\rangle = E_1\langle\psi_2|\psi_1\rangle = 0$

get $E = |c_1|^2E_1 + |c_2|^2E_2$

What condition must c_1 and c_2 satisfy?

Because ψ must be a normalized state: $\langle\psi|\psi\rangle = 1$

$$\langle\psi|\psi\rangle = |c_1|^2\langle\psi_1|\psi_1\rangle + |c_2|^2\langle\psi_2|\psi_2\rangle + c_1^*c_2\langle\psi_1|\psi_2\rangle + c_2^*c_1\langle\psi_2|\psi_1\rangle$$

again because of orthonormality: $\langle\psi|\psi\rangle = |c_1|^2 + |c_2|^2 = 1$

12. In quantum mechanics, we use Hermitian operators because they possess a complete set of orthogonal eigenfunctions with real eigenvalues. This ensures that physical observables like energy, position, or momentum yield real measurement outcomes, and that quantum states can be expanded in terms of the eigenfunctions, providing a consistent framework for prediction and measurement.

Definition of Hermitian operators:

An operator $\hat{A}$ is Hermitian (or self-adjoint) if it satisfies condition: $\int_{-\infty}^{\infty} \psi^* \hat{A} \varphi \, dV = \int_{-\infty}^{\infty} (\hat{A}\psi)^* \varphi \, dV$, for all ψ and φ

Equivalently $\hat{A}^\dagger = \hat{A}$. That is, the operator equals its own adjoint.

Orthonormality of eigenfunctions:

Let $\hat{A}\psi_n = a_n \psi_n$ and $\hat{A}\psi_m = a_m \psi_m$ $a_n, a_m \in \mathbb{R}$

Inner product: $\langle \psi_m | \hat{A}\psi_n \rangle = a_n \langle \psi_m | \psi_n \rangle$

$\langle \hat{A}\psi_m | \psi_n \rangle = a_m \langle \psi_m | \psi_n \rangle$

Since $\hat{A}$ is Hermitian, these must be equal: $a_n \langle \psi_m | \psi_n \rangle = a_m \langle \psi_m | \psi_n \rangle \Rightarrow$

$\Rightarrow (a_n - a_m) \langle \psi_m | \psi_n \rangle = 0$. So if $a_n \neq a_m$ then $\langle \psi_m | \psi_n \rangle = 0$

Thus eigenfunctions corresponding to different eigenvalues are orthogonal

We can normalize the eigenfunctions such that $\langle \psi_n | \psi_n \rangle = 1 \Rightarrow$

$\Rightarrow \langle \psi_n | \psi_m \rangle = \delta_{nm}$. Hence, the eigenfunctions form an orthonormal set.

Completeness:

Well-behaved state ψ in Hilbert space: $\psi = \sum_{n=-\infty}^{\infty} c_n \psi_n$

Using this expansion, we compute the ~~expansion~~ expectation value:

$\langle \psi | \hat{A} | \psi \rangle = \sum_{m,n} c_m^* c_n \langle \psi_m | \hat{A}\psi_n \rangle = \sum_n |c_n|^2 a_n$. Due to orthonormality

$\langle \psi_m, \psi_n \rangle = \delta_{mn}$, the cross terms vanish.

To preserve probability interpretation: $\langle \psi | \psi \rangle = \sum_n |c_n|^2 = 1$

This shows both orthonormality and completeness of the eigenfunctions of Hermitian operators.