Control Questions of Lectures 14 to 16 75

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The interaction of a magnetic dipole moment with an external magnetic field is given by:

$$H' = -\boldsymbol{\mu} \cdot \mathbf{B}$$

For an electron, the magnetic moment is:

$$\boldsymbol{\mu} = -\frac{e}{2m_e}(\mathbf{L} + g_s\mathbf{S}) = -\mu_B \frac{\mathbf{L} + g_s\mathbf{S}}{\hbar}, \quad \mu_B = \frac{e\hbar}{2m_e}$$

Assuming a uniform magnetic field in the z-direction, $\mathbf{B} = B\hat{z}$:

$$H' = \mu_B B \frac{L_z + g_s S_z}{\hbar}$$

Special cases:

- **Normal Zeeman Effect (no spin):** $H' = \mu_B B \frac{L_z}{\hbar}$
- **Anomalous Zeeman Effect (includes spin):** $H' = \mu_B B \frac{L_z + 2S_z}{\hbar}$ (using $g_s \approx 2$)
- **LS-coupling / fine structure:**

$$H' = \mu_B B \, g_J \frac{J_z}{\hbar}$$

where the Landé g-factor is

$$g_J = 1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}, \quad \text{(for LS-coupling)}$$

From time-independent perturbation theory, the first-order energy correction is:

$$\Delta E^{(1)} = \langle \psi^{(0)} | H' | \psi^{(0)} \rangle$$

Depending on the basis, this becomes:

$$\Delta E^{(1)} = \mu_B B \times \begin{cases} m_\ell, & \text{normal Zeeman effect} \\ m_\ell + 2m_s, & \text{anomalous Zeeman effect} \\ g_J m_j, & \text{fine structure states} \end{cases}$$

The first-order correction to the wavefunction is:

$$\left|\psi^{(1)}\right\rangle = \sum_{k \neq n} \frac{\langle k|H'|n\rangle}{E_n^{(0)} - E_k^{(0)}} |k\rangle$$

Since H' is diagonal in the basis $|n\ell m_\ell m_s\rangle$ or $|n\ell s j m_j\rangle$, the matrix elements $\langle k|H'|n\rangle$ vanish for $k \neq n$. Therefore:

$$\Rightarrow \left|\psi^{(1)}\right\rangle = 0$$

Thus, the eigenstates of H_0 remain eigenstates of the total Hamiltonian to first order in B.

The second-order correction to energy is:

$$\Delta E^{(2)} = \sum_{k \neq n} \frac{|\langle k|H'|n\rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

This includes off-diagonal matrix elements and contributes to the quadratic Zeeman effect.

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2 Question 11

Fermi's Golden Rule

The transition rate from an initial state $|i\rangle$ to a final state $|f\rangle$ under a time-dependent perturbation $\hat{H}'$ is given by:

$$W_{i \to f} = \frac{2\pi}{\hbar} \left| \langle f|\hat{H}'|i\rangle \right|^2 \rho(E_f) \qquad \text{(Fermi's Golden Rule)}$$

where:

- $\hat{H}'$ is the perturbation Hamiltonian (e.g., due to an external EM field),
- $\rho(E_f)$ is the density of final states at energy E_f,
- $\langle f|\hat{H}'|i\rangle$ is the transition matrix element.

Harmonic Oscillator in Electromagnetic Field

Unperturbed Hamiltonian:

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Eigenstates: $|n\rangle$ with energies $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$

Perturbation:

$$\hat{H}'(t) = -q\hat{x}E_0\cos(\omega t) = -\frac{qE_0}{2}\hat{x}(e^{i\omega t} + e^{-i\omega t})$$

Matrix Element:

$$\langle n \pm 1|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1} \quad \text{or} \quad \sqrt{n}$$

Selection Rule:

$$\boxed{\Delta n = \pm 1}$$

Transition Probability:

$$W_{n\to n\pm 1} = \frac{2\pi}{\hbar}|-qE_0\langle n \pm 1|\hat{x}|n\rangle|^2\delta(E_{n\pm 1} - E_n \pm \hbar\omega)$$

Hydrogen Atom in Electromagnetic Field

Unperturbed Hamiltonian: Hydrogen atom eigenstates: $|n, \ell, m\rangle$ with energies

$$E_n = -\frac{13.6\,\text{eV}}{n^2}$$

Perturbation (Dipole Approximation):

$$\hat{H}'(t) = -e\hat{\vec{r}}\cdot\vec{E}_0\cos(\omega t) = -\frac{e}{2}\hat{\vec{r}}\cdot\vec{E}_0(e^{i\omega t} + e^{-i\omega t})$$

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Matrix Element:

$$\langle n', \ell', m'|\hat{\vec{r}}|n, \ell, m\rangle$$

Selection Rules for Electric Dipole Transitions:

$$\boxed{\begin{array}{c} \Delta\ell = \pm 1 \\ \Delta m = 0, \pm 1 \\ \Delta n \text{ unrestricted (energy conservation)} \\ \text{Parity must change (i.e., } \ell \to \ell \pm 1) \end{array}}$$

Transition Probability:

$$W_{i\to f} = \frac{2\pi}{\hbar}\left|\langle f|-e\vec{r}\cdot\vec{E}_0|i\rangle\right|^2\delta(E_f - E_i - \hbar\omega)$$

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3 Question 14

Aspect	Schrödinger Equation	Klein–Gordon Equation
Equation	$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$	$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\phi = 0$
Relativity	Non-relativistic. Galilean invariant.	Fully relativistic. Lorentz invariant.
Time Derivatives	First-order in time: requires $\psi(t_0)$.	Second-order in time: requires $\phi(t_0)$ and $\partial_t\phi(t_0)$.
Space Derivatives	Second-order Laplacian ∇^2.	Same: second-order ∇^2 in wave equation form.
Probability Interpretation	$\rho = \|\psi\|^2 \geq 0$	$\rho = \dfrac{i\hbar}{2m}(\phi^*\partial_t\phi - \phi\partial_t\phi^*)$, conserved via $\partial_t\rho + \nabla\cdot\mathbf{j} = 0$, but can be negative
Spin Content	Describes spin-0; spin-$\frac{1}{2}$ requires Pauli or Dirac.	Describes relativistic spin-0 bosons.
Energy Spectrum	Single branch: $E = \frac{p^2}{2m}$, bounded below.	Two branches: $E = \pm\sqrt{p^2c^2 + m^2c^4}$.
Dispersion Relation	$\omega = \frac{\hbar k^2}{2m}$.	$\omega^2 = c^2k^2 + \frac{m^2c^4}{\hbar^2}$.
Lagrangian	No Lorentz-invariant form.	$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2c^2\phi^2$.
Antiparticles	No antiparticles.	Negative energy $\Rightarrow$ antiparticles appear naturally.
Low-Energy Limit	Base non-relativistic quantum theory.	Reduces to Schrödinger for $v \ll c$.
Field Quantization	ψ is single-particle wavefunction.	$\phi \to \hat{\phi}$: needs field quantization.
Typical Use Cases	Atoms, molecules, condensed matter.	Relativistic scalar QFT: pions, Higgs, inflation fields.