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Problem 6: Simultaneous Measurement of $\hat{L}^2$ and $\hat{L}_x$ 20

We need to determine whether the operators $\hat{L}^2$ and $\hat{L}_x$ can be measured simultaneously.

For two observables to be simultaneously measurable, their corresponding operators must commute. Thus, we need to check whether $[\hat{L}^2, \hat{L}_x] = 0$.

Let's calculate this commutator:

$[\hat{L}^2, \hat{L}_x] = \hat{L}^2\hat{L}_x - \hat{L}_x\hat{L}^2$

Since $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, we can write:

$[\hat{L}^2, \hat{L}_x] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_x]$

$= [\hat{L}_x^2, \hat{L}_x] + [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x]$

The first term $[\hat{L}_x^2, \hat{L}_x] = 0$ since an operator commutes with itself and its powers.

For the remaining terms, we need the angular momentum commutation relations: $[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$

For $[\hat{L}_y^2, \hat{L}_x]$, we use the identity $[A^2, B] = A[A, B] + [A, B]A$:

$[\hat{L}_y^2, \hat{L}_x] = \hat{L}_y[\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x]\hat{L}_y$

$= \hat{L}_y(i\hbar\hat{L}_z) + (i\hbar\hat{L}_z)\hat{L}_y$

$= i\hbar(\hat{L}_y\hat{L}_z + \hat{L}_z\hat{L}_y)$

Similarly: $[\hat{L}_z^2, \hat{L}_x] = \hat{L}_z[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x]\hat{L}_z$

$= \hat{L}_z(-i\hbar\hat{L}_y) + (-i\hbar\hat{L}_y)\hat{L}_z$

$= -i\hbar(\hat{L}_z\hat{L}_y + \hat{L}_y\hat{L}_z)$

Therefore: $[\hat{L}^2, \hat{L}_x] = 0 + i\hbar(\hat{L}_y\hat{L}_z + \hat{L}_z\hat{L}_y) - i\hbar(\hat{L}_z\hat{L}_y + \hat{L}_y\hat{L}_z) = 0$

This proves that $[\hat{L}^2, \hat{L}_x] = 0$, meaning that $\hat{L}^2$ and $\hat{L}_x$ can be measured simultaneously. The same would be true for $\hat{L}^2$ and any component $\hat{L}_i$.

Problem 16: Angle Between Angular Momentum Vector and z-axis

In classical mechanics, if we have a vector $\vec{L}$ with a z-component L_z, the angle θ between $\vec{L}$ and the z-axis is given by:

$\cos\theta = \frac{L_z}{|\vec{L}|}$

In quantum mechanics, we cannot specify the exact direction of the angular momentum vector due to the uncertainty principle. However, we can calculate the expectation value of $\cos\theta$:

$\langle\cos\theta\rangle = \frac{\langle\hat{L}_z\rangle}{\sqrt{\langle\hat{L}^2\rangle}}$

For an eigenstate of $\hat{L}^2$ and $\hat{L}_z$ with quantum numbers l and m, we have:

$\hat{L}^2|l, m\rangle = l(l+1)\hbar^2|l, m\rangle$ $\hat{L}_z|l, m\rangle = m\hbar|l, m\rangle$

Therefore: $\langle\cos\theta\rangle = \frac{m\hbar}{\sqrt{l(l+1)}\hbar} = \frac{m}{\sqrt{l(l+1)}}$ 20

For 3d orbitals, $l = 2$.

Case 1: $m = -2$ $\langle\cos\theta\rangle = \frac{-2}{\sqrt{2(2+1)}} = \frac{-2}{\sqrt{6}} = -\frac{2\sqrt{6}}{6} \approx -0.8165$ $\theta \approx \cos^{-1}(-0.8165) \approx 144.7°$

Case 2: $m = +1$ $\langle\cos\theta\rangle = \frac{1}{\sqrt{2(2+1)}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6} \approx 0.4082$ $\theta \approx \cos^{-1}(0.4082) \approx 65.9°$

Problem 26: Ionization Energy Calculation

The ionization energy is the energy required to remove an electron from a bound state to infinity (where the potential energy is zero).

For a hydrogen-like atom with nuclear charge Z, the energy of an electron in state n is:

$E_n = -\frac{Z^2e^2}{8\pi\epsilon_0 a_0} \cdot \frac{1}{n^2} = -\frac{Z^2 Ry}{n^2}$ 20

where $Ry = \frac{e^2}{8\pi\epsilon_0 a_0} \approx 13.6$ eV is the Rydberg energy.

The ionization energy for an electron in state n is:

$I_n = E_\infty - E_n = 0 - (-\frac{Z^2Ry}{n^2}) = \frac{Z^2Ry}{n^2}$

For hydrogen ($Z = 1$):

1. 3s state ($n = 3$, $l = 0$): $I_{3s} = \frac{Ry}{3^2} = \frac{13.6 \text{ eV}}{9} \approx 1.51$ eV
2. 4p state ($n = 4$, $l = 1$): $I_{4p} = \frac{Ry}{4^2} = \frac{13.6 \text{ eV}}{16} \approx 0.85$ eV

For multi-electron atoms, we would need to consider electron-electron interactions and use effective nuclear charge or apply other quantum chemistry methods.

Problem 39: Eigenvalue Problem for Orbital Magnetic Moment

The orbital magnetic moment operator is related to the angular momentum operator by:

$\hat{\vec{\mu}}_l = -\frac{e}{2m_e}\hat{\vec{L}}$

where e is the elementary charge and m_e is the electron mass.

The square of the orbital magnetic moment operator is:

$\hat{\mu}_l^2 = (\frac{e}{2m_e})^2\hat{L}^2$

Since this is proportional to $\hat{L}^2$, they share the same eigenfunctions, which are the spherical harmonics $Y_{lm}(\theta,\phi)$.

The eigenvalue problem is:

$\hat{\mu}_l^2\Psi_{nlm}(\vec{r}) = \mu_l^2\Psi_{nlm}(\vec{r})$

where $\Psi_{nlm}(\vec{r}) = R_{nl}(r)Y_{lm}(\theta,\phi)$ are the hydrogen atom wavefunctions.

The eigenvalues are:

$\mu_l^2 = (\frac{e}{2m_e})^2 l(l+1)\hbar^2 = (\frac{\mu_B}{\hbar})^2 l(l+1)\hbar^2 = \mu_B^2 l(l+1)$

where $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr magneton.

Therefore, the eigenfunctions of $\hat{\mu}_l^2$ are the hydrogen atom wavefunctions $\Psi_{nlm}(\vec{r})$, and the eigenvalues are $\mu_B^2 l(l+1)$.

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Problem 43: Second Order Approximation Equation Proof

We need to prove the equation: $\sum_{k\neq n} |a_k^{(1)}|^2 + (a_n^{(2)*} + a_n^{(2)}) = 0$

This equation appears in time-independent perturbation theory. Let's start with the normalization condition up to second order:

$\langle\psi|\psi\rangle = 1$

Where $|\psi\rangle = |\psi^{(0)}\rangle + \lambda|\psi^{(1)}\rangle + \lambda^2|\psi^{(2)}\rangle + ...$

If we expand $|\psi\rangle$ in terms of the unperturbed eigenstates $\{|n\rangle\}$:

$|\psi\rangle = \sum_k c_k|k\rangle$

where $c_k = c_k^{(0)} + \lambda c_k^{(1)} + \lambda^2 c_k^{(2)} + ...$

Assuming $|\psi^{(0)}\rangle = |n\rangle$, we have $c_k^{(0)} = \delta_{kn}$. 20

The first-order coefficient $c_k^{(1)}$ corresponds to $a_k^{(1)}$ in the problem statement, and similarly $c_k^{(2)}$ corresponds to $a_k^{(2)}$.

The normalization condition gives: $\langle\psi|\psi\rangle = \sum_k |c_k|^2 = 1$

Expanding $|c_k|^2$ to second order: $|c_k|^2 = |c_k^{(0)} + \lambda c_k^{(1)} + \lambda^2 c_k^{(2)}|^2 + O(\lambda^3)$

For $k \neq n$, $c_k^{(0)} = 0$, so: $|c_k|^2 = |\lambda c_k^{(1)} + \lambda^2 c_k^{(2)}|^2 + O(\lambda^3) = \lambda^2|c_k^{(1)}|^2 + O(\lambda^3)$

For $k = n$, $c_n^{(0)} = 1$, so: $|c_n|^2 = |1 + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)}|^2 + O(\lambda^3) = 1 + \lambda(c_n^{(1)*} + c_n^{(1)}) + \lambda^2(|c_n^{(1)}|^2 + c_n^{(2)*} + c_n^{(2)}) + O(\lambda^3)$

Now, the normalization condition becomes: $1 = \sum_{k\neq n} \lambda^2|c_k^{(1)}|^2 + 1 + \lambda(c_n^{(1)*} + c_n^{(1)}) + \lambda^2(|c_n^{(1)}|^2 + c_n^{(2)*} + c_n^{(2)}) + O(\lambda^3)$

Comparing coefficients of λ: $\lambda^1 : c_n^{(1)*} + c_n^{(1)} = 0$ (which means $c_n^{(1)}$ is purely imaginary)

Comparing coefficients of λ^2: $\sum_{k\neq n} |c_k^{(1)}|^2 + |c_n^{(1)}|^2 + c_n^{(2)*} + c_n^{(2)} = 0$

Since $c_n^{(1)}$ is purely imaginary, $|c_n^{(1)}|^2 = -c_n^{(1)*}c_n^{(1)} = 0$

Therefore: $\sum_{k\neq n} |c_k^{(1)}|^2 + c_n^{(2)*} + c_n^{(2)} = 0$

Using the notation from the problem statement: $\sum_{k\neq n} |a_k^{(1)}|^2 + (a_n^{(2)*} + a_n^{(2)}) = 0$

This completes the proof.