Quantum mechanics IV

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1 Is it possible to measure simultaneously the square of angular momentum and its z-projection, i.e., \hat{L}^2 and \hat{L}_z ? Why? Provide proof.(5) 20

The operators \hat{L}^2 and \hat{L}_z commute:

$$[\hat{L}^2, \hat{L}_z] = 0$$

Commuting operators share a complete set of eigenstates. Therefore, both quantities can be simultaneously known with arbitrary precision. Their eigenvalues are:

$$\hat{L}^2 |l,m\rangle = \hbar^2 l(l+1) |l,m\rangle, \quad \hat{L}_z |l,m\rangle = \hbar m |l,m\rangle$$

Thus, simultaneous measurement is allowed.

- 2 A disk with a radius of 1 mm and a mass of 1 mg rotates around an axis perpendicular to its plane with a frequency of 1 Hz. Calculate the value of the orbital quantum number *l* for this spinning disk.(15)
 - Radius $r = 1 \text{ mm} = 1 \times 10^{-3} \text{ m}$
 - Mass $m = 1 \text{ mg} = 1 \times 10^{-6} \text{ kg}$
 - Frequency $f = 1 \,\mathrm{Hz} \Rightarrow \omega = 2\pi \,\mathrm{rad/s}$

Classical angular momentum:

$$L = I\omega = \left(\frac{1}{2}mr^2\right)\omega = \frac{1}{2} \cdot 10^{-6} \cdot (10^{-3})^2 \cdot 2\pi = \pi \cdot 10^{-12} \,\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}$$

Using quantization:

$$L = \hbar \sqrt{l(l+1)} \Rightarrow \sqrt{l(l+1)} = \frac{L}{\hbar} \approx \frac{\pi \times 10^{-12}}{1.055 \times 10^{-34}} \approx 3 \times 10^{22}$$
 units?

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$$l(l+1) \approx 9 \times 10^{44} \Rightarrow l \approx 10^{22}$$

The orbital quantum number is approximately $l \approx 10^{22}$, which reflects the macroscopic nature of the rotation.

3 Show that (for the hydrogen atom) the number of sublevels (degree of degeneration) for an energy level with a fixed value of principal quantum number ncan be calculated with the formula n^2 . Does this calculation take into account the spin effect? (28)

For a given principal quantum number n, the possible values of the orbital quantum number l are:

$$l = 0, 1, 2, \dots, n - 1$$

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For each value of l, the magnetic quantum number m takes 2l + 1 values:

$$m = -l, -(l-1), \dots, 0, \dots, l-1, l$$

The total number of spatial sublevels (degeneracy) is therefore:

$$\sum_{l=0}^{n-1} (2l+1) = 2\sum_{l=0}^{n-1} l + \sum_{l=0}^{n-1} 1 = 2 \cdot \frac{(n-1)n}{2} + n = n(n-1) + n = n^2$$

*not including spin.

4 The electron configuration for a boron atom is written as: $1s^22s^22p^1$. What does this mean? Describe all the numbers in the configuration.(32)

The electron configuration $1s^22s^22p^1$ describes how the electrons of a boron atom are distributed among atomic orbitals. Here's what each part of the notation means:

- $1s^2$: Two electrons are in the first energy level (n = 1), in an s-orbital (l = 0).
- $2s^2$: Two electrons are in the second energy level (n = 2), also in an s-orbital.
- $2p^1$: One electron is in the second energy level (n = 2), in a *p*-orbital (l = 1).

Each component of the notation $n\ell^x$ can be interpreted as:

• n: Principal quantum number — indicates the energy level or shell.

• ℓ : Orbital type — represented by a letter:

$$\ell = 0 \Rightarrow s, \quad \ell = 1 \Rightarrow p, \quad \ell = 2 \Rightarrow d, \quad \ell = 3 \Rightarrow f$$

• Superscript (e.g. the 2 in s^2): Number of electrons occupying that orbital.

Since boron has 5 electrons, this configuration accounts for all of them.

5 Can you prove the following expression (page 127): "In the first-order approximation of λ , the coefficient $a_n^{(1)}$ must satisfy $a_n^{(1)} + (a_n^{(1)})^* = 0$ "?(42)

In time-independent, non-degenerate perturbation theory, the normalized perturbed state is written as:

$$|\psi_n\rangle = |n\rangle + \lambda \sum_{k \neq n} a_k^{(1)} |k\rangle + \lambda a_n^{(1)} |n\rangle + \mathcal{O}(\lambda^2)$$

Here, $a_n^{(1)} \equiv \left(n |\psi_n^{(1)}| n |\psi_n^{(1)} \right)$ represents the component of the first-order correction along $|n\rangle$. Although it has no physical effect (it only alters the phase), we impose normalization:

$$\langle \psi_n | \psi_n | \psi_n \rangle = 1$$
 What does it mean?

Expanding to first order:

$$\langle \psi_n | \psi_n | \psi_n | \psi_n \rangle = \langle n | n | n \rangle + \lambda \left(\left\langle n | \psi_n^{(1)} | n | \psi_n^{(1)} \right\rangle + \left\langle \psi_n^{(1)} | n | \psi_n^{(1)} | n \right\rangle \right) + \mathcal{O}(\lambda^2)$$

This gives:

$$1 + \lambda \left(a_n^{(1)} + \left(a_n^{(1)} \right)^* \right) + \dots = 1$$

Therefore, to preserve normalization:

$$a_n^{(1)} + \left(a_n^{(1)}\right)^* = 0 \Rightarrow \operatorname{Re}\left(a_n^{(1)}\right) = 0 \Rightarrow a_n^{(1)}$$
 is purely imaginary