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7. **Is it possible measure simultaneously absolute value of angular momentum $|\vec{L}|$ and its x projection? Why? Proof.**

   Yes. The operators $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, $\hat{L}_x$ commute:

   $$\left[\hat{L}^2, \hat{L}_x\right] = \left[\hat{L}_x^2, \hat{L}_x\right] + \left[\hat{L}_y^2, \hat{L}_x\right] + \left[\hat{L}_z^2, \hat{L}_x\right].$$

   Since $[\hat{L}_x^2, \hat{L}_x] = 0$ and using commutation relations $[\hat{L}_y, \hat{L}_x] = i\hbar\,\hat{L}_z$, $[\hat{L}_z, \hat{L}_x] = -i\hbar\,\hat{L}_y$:

   $$[\hat{L}_y^2, \hat{L}_x] = \hat{L}_y[\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x]\hat{L}_y = i\hbar\left(\hat{L}_y\hat{L}_z + \hat{L}_z\hat{L}_y\right),$$
   $$[\hat{L}_z^2, \hat{L}_x] = \hat{L}_z[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x]\hat{L}_z = -i\hbar\left(\hat{L}_z\hat{L}_y + \hat{L}_y\hat{L}_z\right),$$

   so that $[\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x] = 0$ and hence $[\hat{L}^2, \hat{L}_x] = 0$. Commuting observables have a common eigenbasis $\{|l, m_x\rangle\}$ with $\hat{L}^2|l, m_x\rangle = \hbar^2\, l(l+1)\,|l, m_x\rangle$ and $\hat{L}_x|l, m_x\rangle = \hbar\, m_x\,|l, m_x\rangle$, so $L = |\vec{L}|$ and L_x can be measured simultaneously.

13. **Write an expression for the x, y and z projection of the angular frequency operators $\hat{\omega}_x$, $\hat{\omega}_y$, and $\hat{\omega}_z$ (the rotating body is a homogeneous sphere of mass M and radius R).**

    In classical mechanics, the angular momentum vector $\vec{L}$ of a rigid body is related to its angular velocity $\vec{\omega}$ by the inertia tensor $\vec{I}$:

    $$\vec{L} = \vec{I}\,\vec{\omega}.$$

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    For a homogeneous sphere the inertia tensor becomes scalar,

    $$I = \int_V (r\sin\theta)^2 \rho r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \rho\int_0^R\int_0^\pi\int_0^{2\pi} r^4\sin^3\theta\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r =$$
    $$= \underbrace{\rho}_{=M/V} \cdot 2\pi\cdot 4/3\cdot R^5/5 = \frac{3M\cdot 2\pi\cdot 4\cdot R^5}{4\pi R^3\cdot 3\cdot 5} = \frac{2}{5}MR^2,$$

    so that $\hat{\vec{L}} = I\,\vec{\omega}$. In quantum mechanics, the corresponding classical quantities are represented by operators: $\hat{\vec{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ and $\hat{\vec{\omega}} = (\hat{\omega}_x, \hat{\omega}_y, \hat{\omega}_z)$. Since $\vec{I} = I\,\vec{1}$, it follows that $\hat{\vec{\omega}} = \frac{\hat{\vec{L}}}{I}$. Hence, each component of the angular velocity operator is proportional to the corresponding component of the angular momentum operator:

    $$\hat{\omega}_x = \frac{\hat{L}_x}{I}, \quad \hat{\omega}_y = \frac{\hat{L}_y}{I}, \quad \hat{\omega}_z = \frac{\hat{L}_z}{I}.$$

20. **Obtain the equation for radial part of wave function:**

    $$-\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right)\right] + \left[\frac{\hbar^2\, l(l+1)}{2M\, r^2} + U(r)\right] R_{nl}(r) = E\, R_{nl}(r),$$

    Starting from the time–independent Schrödinger equation:

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    $$-\frac{\hbar^2}{2M}\,\nabla^2\psi(\vec{r}) + U(r)\,\psi(\vec{r}) = E\,\psi(\vec{r}).$$

    In spherical coordinates:

    $$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\,\hat{L}^2_{\theta\phi}.$$

Separaating the variables:

$$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)\,Y_{\ell m}(\theta,\phi),$$

where $Y_{\ell m}$ are spherical harmonics and $R_{n\ell}(r)$ is purely radial. Substituting into the full equation:

$$-\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\,\hat{L}^2_{\theta\phi}(R\,Y)\right] + U(r)\,R\,Y = E\,R\,Y.$$

Separating radial and angular parts:

$$\frac{\partial}{\partial r} = Y\,R', \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = Y\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2R'\right),$$

and $\hat{L}^2_{\theta\phi}(R\,Y) = R\,\hat{L}^2_{\theta\phi}Y$. Using the eigenvalue equation for spherical harmonics:

$$\hat{L}^2_{\theta\phi}Y_{\ell m} = \hbar^2\,\ell(\ell+1)\,Y_{\ell m}, \quad \text{so } \hat{L}^2_{\theta\phi}(R\,Y) = \hbar^2\,\ell(\ell+1)\,R\,Y.$$

Substituting and dividing by Y:

$$-\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2R'\right) - \frac{\ell(\ell+1)}{r^2}R\right] + U(r)\,R = E\,R.$$

Rearranging to the standard form:

$$-\frac{\hbar^2}{2M}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)\right] + \left[\frac{\hbar^2\,\ell(\ell+1)}{2M\,r^2} + U(r)\right]R(r) = E\,R(r).$$

36. **How looks like the electron configuration for Li, Na and K atoms? Why? What do all these materials have in common (in terms of physical properties)? How does this relate to the configuration of the electrons?**

- **Lithium (Li, $Z = 3$)**: $1s^2\,2s^1$.
- **Sodium (Na, $Z = 11$)**: $1s^2\,2s^2\,2p^6\,3s^1$.
- **Potassium (K, $Z = 19$)**: $1s^2\,2s^2\,2p^6\,3s^2\,3p^6\,4s^1$.

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Reasons:

- Principle of minimum energy. Electrons fill orbitals in order of increasing energy: $E_{1s} < E_{2s} < E_{2p} < E_{3s} < \dots$
- Exclusion principle. Each spatial orbital, specified by the quantum numbers, can accommodate at most two electrons, and these two must have opposite spin quantum numbers: $m_s = +\frac{1}{2}$ and $m_s = -\frac{1}{2}$.
- Screening and effective nuclear charge. Core electrons generate a spherically symmetric potential that partially screens the nuclear charge. As a result, the valence electron experiences an effective charge Z_{eff} and moves in a central potential whose energy levels follow the same ordering derived from the radial Schrödinger equation.

From this it follows that an electron will occupy the first available (lowest-energy) s-orbital of principal quantum number n, yielding the configuration ns^1.

Consequences (because of the common valence configuration ns^1):

- Low first ionization energy: the lone s–electron is only weakly bound.
- Formation of M^+ cations: they readily lose one electron.
- Characteristic group–1 ("alkali metal") properties: high chemical reactivity, metallic luster, softness, low melting points and high electrical conductivity.

45. **Why for the second-order energy correction of a harmonic oscillator in a constant force field (p. 129) we need to take into account only two terms of the sum n, $n+1$ and n, $n-1$.**

The perturbation by a constant force F is

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$$\hat{H}' = F\,\hat{x}.$$

Expressing the position operator in terms of creation and annihilation operators:

$$\hat{x} = \sqrt{\frac{\hbar}{2M\omega}}\,(a + a^\dagger).$$

The perturbation matrix has nonzero elements only between adjacent levels:

$$\langle k|\hat{H}'|n\rangle = F\sqrt{\frac{\hbar}{2M\omega}}\,\big(\langle k|\,a\,|n\rangle + \langle k|\,a^\dagger\,|n\rangle\big)$$

Since

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,,$$

It follows that

$$\langle k|\hat{H}'|n\rangle \neq 0 \quad \Longrightarrow \quad k = n \pm 1.$$

follows from what?

Therefore, the second–order correction

$$E_n^{(2)} = \sum_{k\neq n} \frac{\left|\langle k|\,\hat{H}'\,|n\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}} = \frac{|\,\langle n-1|\,\hat{H}'\,|n\rangle\,|^2}{E_n^{(0)} - E_{n-1}^{(0)}} + \frac{|\,\langle n+1|\,\hat{H}'\,|n\rangle\,|^2}{E_n^{(0)} - E_{n+1}^{(0)}}.$$

thus contains contributions only from $k = n+1$ and $k = n-1$, since for all other k the numerator vanishes.