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6. Is it possible measure simultaneously square of angular momentum and its x projection?

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Total angular-momentum operator

$$L^2 = L_x^2 + L_y^2 + L_z^2.$$

Two observables A and B can be measured simultaneously with definite values if and only if

$$[A, B] = 0.$$

$[L^2, L_x] = 0$

Starting from $L^2 = L_x^2 + L_y^2 + L_z^2$, we have

$$[L^2,\ L_x] = [L_x^2, L_x]\ +\ [L_y^2, L_x]\ +\ [L_z^2, L_x].$$

$$[L_x^2, L_x] = L_x[L_x, L_x] + [L_x, L_x]L_x = 0 + 0 = 0.$$

Using the identity $[A^2, B] = A[A, B] + [A, B]A$:

$$[L_y^2, L_x] = L_y[L_y, L_x] + [L_y, L_x]\, L_y.$$

From $[L_y, L_x] = i\hbar\,\varepsilon_{yxk}L_k = -\,i\hbar\,L_z$, it follows that

$$[L_y^2, L_x] = -\,i\hbar\,(L_yL_z + L_zL_y).$$

Similarly,

$$[L_z^2, L_x] = L_z[L_z, L_x] + [L_z, L_x]\, L_z,$$

and since $[L_z, L_x] = i\hbar\,\varepsilon_{zxk}L_k = i\hbar\,L_y$, we get

$$[L_z^2, L_x] = +\,i\hbar\,(L_zL_y + L_yL_z).$$

Adding these contributions,

$$[L^2, L_x] = 0\ -\ i\hbar\,(L_yL_z + L_zL_y)\ +\ i\hbar\,(L_zL_y + L_yL_z) = 0.$$

Since $[L^2, L_x] = 0$, the operators L^2 and L_x commute and therefore can be measured simultaneously. A common set of eigenstates $\{|l, m_x\rangle\}$ satisfies

$$L^2\,|l, m_x\rangle = \hbar^2\,l(l+1)\ |l, m_x\rangle\,, \quad L_x\,|l, m_x\rangle = \hbar\,m_x\ |l, m_x\rangle\,,$$

where $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $m_x = -l, -l+1, \ldots, +l$.

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19. How to calculate in quantum mechanics the kinetic energy of a rotating body with the moment of inertia I ?

Kinetic Energy of a Quantum Rigid Rotor

In quantum mechanics, a *rigid rotor* of moment of inertia I has kinetic-energy operator

$$\hat{T}\ =\ \frac{\hat{L}^2}{2I},$$

where

- $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ is the total angular-momentum operator,

- I is the moment of inertia about the rotation axis.

The eigenvalues of $\hat{L}^2$ are

$$\hat{L}^2\,|l,m\rangle = \hbar^2\,l(l+1)\;|l,m\rangle\,, \quad l=0,1,2,\ldots, \quad m=-l,-l+1,\ldots,+l.$$

Hence the energy levels of the rigid rotor are

$$E_l = \frac{\hbar^2\,l(l+1)}{2I}.$$

For a point mass m constrained to move on a circle of radius r, the moment of inertia about the center is

$$I \;=\; m\,r^2.$$

Substituting $I = mr^2$ into the expression for E_l gives

$$E_l = \frac{\hbar^2\,l(l+1)}{2\,m\,r^2}.$$

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The *ground state* ($l=0$) has $E_0 = 0$. The *first excited state* ($l=1$) has

$$E_1 = \frac{\hbar^2\cdot 1\cdot 2}{2\,m\,r^2} = \frac{\hbar^2}{m\,r^2}.$$

Classically,

$$T = \frac{1}{2}\,m_e\,v^2.$$

Equate this to E_1:

$$\frac{1}{2}\,m_e\,v^2 \;=\; \frac{\hbar^2}{m_e\,r^2} \quad\Longrightarrow\quad v = \sqrt{\frac{2\,\hbar^2}{m_e^2\,r^2}} = \sqrt{2}\,\frac{\hbar}{m_e\,r}.$$

Substituting numerical values:

$$\hbar = 1.054\times10^{-34}\,\mathrm{J\,s}, \quad m_e = 9.109\times10^{-31}\,\mathrm{kg}, \quad r = 1.0\times10^{-10}\,\mathrm{m},$$

we find

$$v \;\approx\; \sqrt{2}\,\frac{1.054\times10^{-34}}{(9.109\times10^{-31})\,(1.0\times10^{-10})} \;\approx\; 1.6\times10^{6}\,\mathrm{m/s}.$$

23. Write the radial part of wave function Rnl for quantum numbers n=3 and l=2

In a hydrogen-like atom the radial part of the bound-state wavefunction can be written in the form

$$R_{n\ell}(r) = N_{n\ell}\,\rho^{\,\ell}\,e^{-\rho/2}\,L^{2\ell+1}_{n-\ell-1}(\rho), \qquad \rho = \frac{2r}{n\,r_0},$$

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$n=3,\ \ell=2$

Here

$$n-\ell-1 = 3-2-1 = 0 \quad\Longrightarrow\quad L_0^5(\rho) = 1.$$

Hence the *unnormalized* radial part is

$$R_{3,2}(r) \;\propto\; \rho^2\,e^{-\rho/2} \;=\; \left(\tfrac{2r}{3r_0}\right)^2 \exp\!\left(-\tfrac{r}{3r_0}\right).$$

therefore set

$$R_{3,2}(r) = C\left(\tfrac{r}{r_0}\right)^2 \exp\!\left(-\tfrac{r}{3r_0}\right),$$

with C to be determined.

The full three-dimensional normalization requires

$$\int_0^\infty R_{3,2}^2(r)\, r^2\, dr = 1.$$

Substituting $r = r_0 x$, so $dr = r_0\, dx$ and $r^2\, dr = r_0^3\, x^2\, dx$. Then

$$\int_0^\infty \left| C\, x^2 e^{-x/3} \right|^2 r_0^3 x^2\, dx = 1 \implies |C|^2\, r_0^3 \int_0^\infty x^6\, e^{-2x/3}\, dx = 1.$$

The remaining integral is evaluated via the Gamma function:

$$\int_0^\infty x^6\, e^{-2x/3}\, dx = 6! \left(\frac{3}{2}\right)^7 = \frac{98415}{8}.$$

Hence

$$|C|^2\, r_0^3\, \frac{16}{81}\, \frac{98415}{8} \;=\; 1 \quad\Longrightarrow\quad |C|^2 = \frac{1}{2430\, r_0^3}.$$

Taking the positive square root,

$$C = \frac{2\sqrt{2}}{81\sqrt{15}\, r_0^3}.$$

$$\boxed{R_{3,2}(r) = \frac{2\sqrt{2}}{81\sqrt{15}\, r_0^3} \left(\tfrac{r}{r_0}\right)^2 \exp\!\left(-\tfrac{r}{3\, r_0}\right).}$$

39.What does the eigenvalue problem look like for the operator of square of the orbital magnetic moment of an electron in a hydrogen atom?

In a hydrogen-like atom the orbital magnetic moment operator of the electron is defined by

$$\hat{\boldsymbol{\mu}}_\ell = -\frac{e}{2m}\, \hat{\mathbf{L}},$$

where

- $e > 0$ is the elementary charge,
- m is the electron mass,
- $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ is the orbital angular-momentum operator.

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We are interested in the square of this operator,

$$\hat{\mu}_\ell^2 = \hat{\boldsymbol{\mu}}_\ell \cdot \hat{\boldsymbol{\mu}}_\ell = \left(-\frac{e}{2m}\right)^2 \hat{L}^2 = \frac{e^2}{4m^2}\, \hat{L}^2 = \frac{\mu_B^2}{\hbar^2}\, \hat{L}^2,$$

where

$$\mu_B = \frac{e\hbar}{2m} \quad \text{(Bohr magneton)}.$$

The eigenvalue problem for $\hat{\mu}_\ell^2$ is

$$\hat{\mu}_\ell^2\, \Psi(\mathbf{r}) = \lambda\, \Psi(\mathbf{r}).$$

Substituting $\hat{\mu}_\ell^2 = (\mu_B^2/\hbar^2)\, \hat{L}^2$ gives

$$\frac{\mu_B^2}{\hbar^2}\, \hat{L}^2\, \Psi = \lambda\, \Psi \quad\Longrightarrow\quad \hat{L}^2\, \Psi = \frac{\hbar^2}{\mu_B^2}\, \lambda\, \Psi.$$

We know that $\hat{L}^2$ and $\hat{L}_z$ admit a common basis of eigenfunctions, the spherical harmonics $Y_{\ell m}(\theta,\phi)$, which satisfy

$$\hat{L}^2\, Y_{\ell m} = \hbar^2\, \ell(\ell+1)\, Y_{\ell m}, \quad \hat{L}_z\, Y_{\ell m} = \hbar\, m\, Y_{\ell m},$$

with quantum numbers

$$\ell = 0, 1, 2, \ldots, \quad m = -\ell, -\ell+1, \ldots, +\ell.$$

The stationary eigenstates of the hydrogen atom separate in spherical coordinates as

$$\Psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)\; Y_{\ell m}(\theta,\phi),$$

where $R_{n\ell}(r)$ is the well-known radial function (solution of the radial Schrödinger equation). Because $\hat{\mu}_\ell^2$ acts only on the angular variables, we have

$$\hat{\mu}_\ell^2\, \Psi_{n\ell m} = \frac{\mu_B^2}{\hbar^2}\, \hat{L}^2 \left[R_{n\ell}(r)\, Y_{\ell m}(\theta,\phi)\right] = \frac{\mu_B^2}{\hbar^2}\, \left(\hbar^2\, \ell(\ell+1)\right) R_{n\ell}(r)\, Y_{\ell m}(\theta,\phi).$$

Comparing with the eigenvalue equation $\hat{\mu}_\ell^2 \Psi = \lambda \Psi$ gives

$$\lambda \;=\; \mu_B^2\, \ell(\ell+1), \quad \ell = 0, 1, 2, \ldots$$

Each eigenvalue $\lambda_\ell = \mu_B^2 \ell(\ell+1)$ is $(2\ell+1)$-fold degenerate in the magnetic quantum number m.

42.Can you prove the following expression (page 127): "In the first order approximation of coefficient...

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We consider a Hamiltonian of the form

$$\hat{H}(\lambda) \;=\; \hat{H}^{(0)} \;+\; \lambda\, \hat{H}',$$

and seek its eigenvalues and eigenfunctions expanded in powers of the small parameter λ:

$$E_n(\lambda) = E_n^{(0)} \;+\; \lambda\, E_n^{(1)} \;+\; \mathcal{O}(\lambda^2),$$
$$\Psi_n(\lambda) = \Psi_n^{(0)} \;+\; \lambda\, \Psi_n^{(1)} \;+\; \mathcal{O}(\lambda^2).$$

The unperturbed states $\{\Psi_m^{(0)}\}$ form a complete orthonormal set: $\langle \Psi_m^{(0)} | \Psi_k^{(0)} \rangle = \delta_{mk}$.

Inserting these expansions into $\hat{H}(\lambda)\, \Psi_n(\lambda) = E_n(\lambda)\, \Psi_n(\lambda)$ and collecting terms of order λ^0 and λ^1 gives:

- *Zeroth order* (λ^0):
$$\hat{H}^{(0)}\, \Psi_n^{(0)} = E_n^{(0)}\, \Psi_n^{(0)}.$$
- *First order* (λ^1):
$$\left(\hat{H}^{(0)} - E_n^{(0)}\right) \Psi_n^{(1)} + \hat{H}'\, \Psi_n^{(0)} = E_n^{(1)}\, \Psi_n^{(0)}.$$

We expand the first-order correction in the unperturbed basis:

$$\Psi_n^{(1)} = \sum_m a_m^{(1)}\, \Psi_m^{(0)}, \qquad a_m^{(1)} = \left\langle \Psi_m^{(0)} \;\middle|\; \Psi_n^{(1)} \right\rangle.$$

Projecting the first-order equation onto $\langle \Psi_k^{(0)}|$. For $k = n$ one finds

$$E_n^{(1)} = \left\langle \Psi_n^{(0)} \middle| \hat{H}' \middle| \Psi_n^{(0)} \right\rangle,$$

the familiar diagonal energy shift. For $k \neq n$,

$$a_k^{(1)}\, (E_k^{(0)} - E_n^{(0)}) = \left\langle \Psi_k^{(0)} \middle| \hat{H}' \middle| \Psi_n^{(0)} \right\rangle,$$

hence

$$a_k^{(1)} = \frac{\langle \Psi_k^{(0)} | \hat{H}' | \Psi_n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}}.$$

At all orders the perturbed state must remain normalized:

$$\langle \Psi_n(\lambda) \mid \Psi_n(\lambda) \rangle = 1.$$

Expanding to first order in λ:

$$\begin{aligned} 1 &= \left\langle \Psi_n^{(0)} + \lambda\, \Psi_n^{(1)} \;\middle|\; \Psi_n^{(0)} + \lambda\, \Psi_n^{(1)} \right\rangle + \mathcal{O}(\lambda^2) \\ &= 1 + \lambda \Big[\langle \Psi_n^{(1)} | \Psi_n^{(0)} \rangle + \langle \Psi_n^{(0)} | \Psi_n^{(1)} \rangle \Big] + \mathcal{O}(\lambda^2). \end{aligned}$$

Since the λ^0 term already equals 1, the coefficient of λ must vanish:

$$\langle \Psi_n^{(1)} | \Psi_n^{(0)} \rangle \;+\; \langle \Psi_n^{(0)} | \Psi_n^{(1)} \rangle = 0.$$

But by definition $\langle \Psi_n^{(0)} | \Psi_n^{(1)} \rangle = a_n^{(1)}$ and its complex conjugate is $\langle \Psi_n^{(1)} | \Psi_n^{(0)} \rangle = \left(a_n^{(1)}\right)^*$.

$$\boxed{a_n^{(1)} \;+\; \left(a_n^{(1)}\right)^* \;=\; 0,}$$