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Control Questions Lectures 10 to 13

Karl Kukk

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1 Question 8

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$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2$$

Recall that the square of the angular momentum operator is:

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

and that $\hat{L}_i$ (for $i = x, y, z$) involve first-order derivatives.

∇^2 in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{\hbar^2 r^2}$$

Thus, the Hamiltonian becomes:

$$\hat{H} = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{\hbar^2 r^2}\right]$$

$\hat{H}$ contains $\hat{L}^2$ explicitly, and both $\hat{H}$ and $\hat{L}^2$ are functions of position and momentum operators. Because $\hat{L}^2$ commutes with itself and its components commute with the radial part of the Hamiltonian, which acts only on r, we get:

$$[\hat{H}, \hat{L}^2] = 0$$

The commutation relation $[\hat{H}, \hat{L}^2] = 0$ implies that $\hat{H}$ and $\hat{L}^2$ share a common set of eigenfunctions. Therefore, angular momentum squared is a conserved quantity in this system and therefore the system has rotational symmetry. This conservation is a consequence of Noether's theorem, which states that every continuous symmetry of the action of a physical system with conservative forces has a corresponding conservation law.

The Heisenberg Uncertainty Principle states that non-commuting observables cannot be simultaneously known with arbitrary precision:

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$$

Since $[\hat{H}, \hat{L}^2] = 0$, energy and angular momentum squared can be simultaneously measured with arbitrary precision. Thus, there is no uncertainty relation between them, and we can have quantum states that are eigenstates of both $\hat{H}$ and $\hat{L}^2$.

2 Question 13

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For a solid sphere rotating in space, the moment of inertia tensor is given by:

$$I = \frac{2}{5}MR^2$$

Angular momentum $\vec{L}$ and angular velocity $\vec{\omega}$ are related via:

$$\vec{L} = I\vec{\omega} \quad \Rightarrow \quad \vec{\omega} = \frac{\vec{L}}{I}$$

Therefore, the components of the angular frequency operators are:

$$\begin{aligned}
\hat{\omega}_x &= \frac{\hat{L}_x}{I} = \frac{5}{2MR^2}\hat{L}_x \\
\hat{\omega}_y &= \frac{\hat{L}_y}{I} = \frac{5}{2MR^2}\hat{L}_y \\
\hat{\omega}_z &= \frac{\hat{L}_z}{I} = \frac{5}{2MR^2}\hat{L}_z
\end{aligned}$$

3 Question 27

The total energy of the hydrogen atom in the Bohr model is:

$$E_n = -\frac{13.6\,\text{eV}}{n^2}$$

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For $n = 3$:

$$E_3 = -\frac{13.6}{3^2} = -\frac{13.6}{9} \approx -1.51\,\text{eV}$$

In the Bohr model, classical relations hold and the potential and kinetic energies are $U = 2E$ and $K = -E$ respectively

Therefore:

$$K_3 = -E_3 = 1.51\,\text{eV}$$
$$U_3 = 2E_3 = -3.02\,\text{eV}$$

- Total energy $E_3 = -1.51\,\text{eV}$
- Kinetic energy $K_3 = 1.51\,\text{eV}$
- Potential energy $V_3 = -3.02\,\text{eV}$

4 Question 31

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For a 4p orbital the principal quantum number is $n = 4$ and the azimuthal quantum number is $\ell = 1$ (p-orbital), the magnetic quantum numbers are $m_\ell = -1, 0, +1$, giving us 3 possible orbitals. Each orbital can accommodate 2 electrons, one with negative, and one with positive spin.

$$\text{Maximum number of electrons in 4p} = 3 \times 2 = 6$$

For a 2d orbital the principal quantum number is $n = 2$. For $n = 2$, allowed values of ℓ are 0 (s) and 1 (p) and therefore a $\ell = 2$ (d-orbital) is not allowed for $n = 2$.

$$\text{Therefore, the 2d orbital does not exist} \Rightarrow 0 \text{ electrons}$$

5 Question 42

$$|\psi(t)\rangle = \left(1 + \lambda a_n^1(t)\right)|n\rangle + \lambda \sum_{m \neq n} a_m^1(t)\,|m\rangle + \mathcal{O}(\lambda^2)$$

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The system starts at $|n\rangle$. $a_n^1(t)$ and $a_m^1(t)$ are first-order coefficients.

We normalize the wavefunction:

$$\langle \psi(t) | \psi(t) \rangle = 1.$$

We compute the inner product up to first order in λ:

$$\begin{aligned}
\langle \psi(t) | \psi(t) \rangle &= \left(\langle n| + \lambda a_n^{1*}(t) \langle n| + \lambda \sum_{k \neq n} a_k^{1*}(t) \langle k| \right) \left(|n\rangle + \lambda a_n^{1}(t) |n\rangle + \lambda \sum_{m \neq n} a_m^{1}(t) |m\rangle \right) + \mathcal{O}(\lambda \\
&= \langle n|n \rangle + \lambda a_n^1(t) \langle n|n \rangle + \lambda a_n^{1*}(t) \langle n|n \rangle + \lambda \sum_{m \neq n} a_m^1(t) \langle n|m \rangle + \lambda \sum_{k \neq n} a_k^{1*}(t) \langle k|n \rangle + \mathcal{O}(
\end{aligned}$$

Because of orthonormality $\langle n|n \rangle = 1, \qquad \langle n|m \rangle = \langle k|n \rangle = 0 \quad \text{for } m, k \neq n$

Then we calculate:

$$\langle \psi(t) | \psi(t) \rangle = 1 + \lambda \left(a_n^1(t) + a_n^{1*}(t) \right) + \mathcal{O}(\lambda^2).$$

To satisfy the normalization condition to order λ, we must require that the coefficient of λ vanishes and therefore:

$$a_n^1(t) + \left(a_n^1(t) \right)^* = 0.$$